6 January 2015
Time: 9:00-11:00

MATH 225
FINAL

SURNAME

NAME

STUDENT NO

SIGNATURE

Department

IMPORTANT:
- This exam consists of 5 questions of equal weight.
- Please read the questions carefully and write your answers under the corresponding question. Be neat.
- Show all your work. Correct answers without sufficient explanation might not get full grades.
- Calculators and dictionaries are not allowed.
- Close your cellular phones.

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1) Consider the differential equation \( y(x^3 - y)dx - x(x^3 + y)dy = 0 \).

a) (5 pts.) Find a real number \( \alpha \) so that \( \rho(x, y) = \frac{x^\alpha}{y^\beta} \) is an integrating factor of this equation. Show all your work. Correct answers without sufficient explanation might not get full grades.

\[
\frac{x^3}{y^3} \left( \frac{x}{y} - y \right) dx + \frac{x^4}{y^3} \left( -x^4 - xy \right) dy = 0
\]

\[
\frac{\partial M}{\partial y} = -2 \frac{x^3+y}{y^3} + \frac{x}{y^2} = \frac{\partial N}{\partial x} = \frac{-x^4+y^3+2x^3}{y^3} - \frac{x^3+y}{y^2}
\]

\[
\therefore -2 = -(4+\alpha) \quad \Rightarrow \quad \alpha = -2
\]

b) (10 pts.) For this \( \alpha \), solve the given differential equation which satisfies \( y(1) = -1 \). Show all your work. Correct answers without sufficient explanation might not get full grades.

When \( \alpha = -2 \):

\[
\left( \frac{x}{y^2} - \frac{1}{xy} \right) dx + \left( -\frac{x^2}{y^3} - \frac{1}{xy^2} \right) dy = 0 \quad (2)
\]

\[
\frac{\partial M}{\partial y} = -2 \frac{x}{y^3} + \frac{1}{x^2 y^2} = \frac{\partial N}{\partial x} = -2 \frac{x}{y^3} + \frac{1}{x^2 y^2} \quad \Rightarrow \quad (2) \text{ is exact.}
\]

\[
F(x, y) = \int \left( \frac{x}{y^2} - \frac{1}{xy} \right) dx = \frac{x^2}{2y^2} + \frac{1}{xy} + g(y)
\]

\[
\frac{\partial F}{\partial y} = \frac{-x^2}{y^3} + \frac{1}{xy} + g'(y) = N(x, y) = -\frac{x^2}{y^3} - \frac{1}{xy^2} \quad \Rightarrow \quad g'(y) = 0 \quad \Rightarrow \quad g(y) = C.
\]

\[
\frac{x^2}{y^2} + \frac{1}{xy} = C = 0
\]

\[
y(1) = -1 \quad \Rightarrow \quad \frac{1}{2} - 1 + C = 0 \quad \Rightarrow \quad C = \frac{1}{2}
\]

\[
\frac{x^2}{2y^2} + \frac{1}{xy} + \frac{1}{2} = 0
\]
2) (20 pts.) Find a 4x5 reduced echelon matrix $R$ with all the following properties:

a) $\text{rank}(R) = 3$,

b) the leading entries of the first two rows appear in the first two columns,

c) the 4x4 non homogeneous system with augmented matrix $R$ is inconsistent,

d) $(2,1,1,0,0)$ and $(-1,0,2,-1,0)$ are solutions of the homogeneous system with coefficient matrix $R$.

THERE IS NO PARTIAL CREDIT FOR THIS QUESTION. YOU WILL GET EITHER 0 OR 20.

Show all your work. Correct answers without sufficient explanation might not get full grades.

Using a, b, c and d, use d1:

$$ R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} $$

To find a, b, c and d, use d1:

$$ \begin{bmatrix} 1 & 0 & a & b & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2t + a \\ 1 + c \\ 0 \\ 0 \end{bmatrix} \Rightarrow a = -2, c = -1 $$

$$ \begin{bmatrix} 1 & -2 & b & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 - 2b \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow b = -5 $$

$$ \begin{bmatrix} 1 & -1 & -d & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 - d - b \\ -2 - d \\ 0 \\ 0 \end{bmatrix} \Rightarrow d = -2 $$

$$ R = \begin{bmatrix} 1 & -2 & -5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$
3) a) (10 pts.) Is there a 2x5 matrix A whose null space is

\[ \text{Null}(A) = \left\{ (a, b, c, d, e) \in \mathbb{R}^5 \mid a = 3b, c = d = e \right\} \]?

Explain your answer.

Show all your work. Correct answers without sufficient explanation might not get full grades.

\[ \text{Null}(A) = \left\{ (3b, b, c, d, e) \mid b, c, d, e \in \mathbb{R} \right\} = \text{span}(v_1, v_2) \]

Thus \( \beta_1, \beta_2 \) is a basis for \( \text{Null}(A) \). Nullity = 2.

\[ \text{rank}(A) + \text{null}(A) = 5 \] but rank of a 2x5 matrix can be at most 2.

So there is no such a matrix A.

b) (10 pts.) Let \( A \) be an \( n \times n \) matrix, and let \( \alpha_1, \alpha_2, \ldots, \alpha_k \) be a collection of linearly independent vectors in \( \mathbb{R}^n \). Let \( \beta_1, \beta_2, \ldots, \beta_k \) be vectors which satisfy \( \beta_j = A\alpha_j \) for \( j = 1, 2, \ldots, k \). Show that the \( \beta \)'s must be linearly independent if \( A \) is non-singular.

Show all your work. Correct answers without sufficient explanation might not get full grades.

Write \( c_1\beta_1 + c_2\beta_2 + \ldots + c_k\beta_k = 0 \) (I)

Aim: To show that \( c_1 = c_2 = \ldots = c_k = 0 \) is the only solution.

We are given \( \beta_j = A\alpha_j \), \( j = 1, 2, \ldots, k \). Hence eqn. (I) becomes,

\[ c_1(A\alpha_1) + c_2(A\alpha_2) + \ldots + c_k(A\alpha_k) = 0. \]

Since \( A \) is invertible, we multiply both sides by \( A^{-1} \) to get

\[ c_1\alpha_1 + c_2\alpha_2 + \ldots + c_k\alpha_k = 0 \] \( \Rightarrow \) \( c_1 = c_2 = \ldots = c_k = 0 \) is the only solution since \( \alpha_1, \ldots, \alpha_k \) are lin. independent.

Thus, \( \beta_1, \beta_2 \) are lin. indep.

Note: Here you cannot say that either \( \alpha \) or \( c_1\alpha_1 + c_2\alpha_2 + c_k\alpha_k = 0 \) because for two matrices \( A \) (\( B \)), \( AB \) may be zero, but \( B \) may not be zero & \( B \) may not be zero.
(5 pts. each) Suppose $A$ is a $4 \times 4$ matrix and $v_1, v_2, v_3, v_4$ are non-zero vectors in $\mathbb{R}^4$ so that $Av_1 = 2v_1$, $Av_2 = -v_2$, $Av_3 = v_3$, $Av_4 = -2v_4$. Answer the following questions.

Show all your work. Correct answers without sufficient explanation might not get full grades.

2.1

$a)$ Find the characteristic polynomial of $A$.

$$p(\lambda) = (\lambda - 2)(\lambda - 1)(\lambda + 1)(\lambda + 2) = (\lambda^2 - 5)(\lambda^2 - 1) = \lambda^4 - 6\lambda^2 + 4$$

$b)$ Is $A$ diagonalizable? If so, find a diagonal matrix $D$ such that $P^{-1}AP = D$. If not so, explain why not.

Yes, $A$ is diagonalizable because it's a $4 \times 4$ matrix with 4 distinct eigenvalues.

$$D = \text{diag}(2, -1, 1, -2)$$

c) Is $A$ invertible? If so, find $A^{-1}$ in terms of $A$. If not so, explain why not.

Yes, $A$ is invertible because $\det A = 3 \cdot 2 \cdot 2 \cdot 4 = 48 \neq 0$.

By Cayley-Hamilton Thm, $A$ satisfies its characteristic polynomial, i.e.,

$$A^4 - 5A + 4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = 0 \Rightarrow A (A^3 - 5A) = -4I = A \left( -\frac{1}{4}A^3 + \frac{5}{4}A \right) = I$$

Thus, $A^{-1} = -\frac{1}{4}A^3 + \frac{5}{4}A$.

d) Find $A^4(2v_2 - 3v_3)$ in terms of $v_2$ and $v_3$.

$$A^4 = A^3A = A^2(Av_1) = A^2(2v_1) = 2A^2v_1 = 2Av_2 = 4v_2$$

$$A^4 = A^3A = A^2(Av_2) = A^2(-v_2) = -A^2v_2 = -2A^2v_2 = -4v_2$$

Thus, $A^4(2v_2 - 3v_3) = 2(4v_2) - 3(-4v_2) = 2v_2 - 3v_3$. 

$$A^4(2v_2 - 3v_3) = 2(4v_2) - 3(-4v_2) = 2v_2 - 3v_3.$$
5) Decode the message \text{VOJEWRROVTWBYE} given that it is a Hill cipher with enciphering matrix \[
\begin{bmatrix}
5 & 0 \\
0 & 1 \\
\end{bmatrix}
\].

Use the following tables, if necessary.

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\[
A = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } A^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mod 26
\]

\[
\text{VOJEWRROVTWBYE} \equiv \begin{bmatrix} 22 & 15 \\ 25 & 15 \end{bmatrix} \mod 26
\]

\[
\begin{bmatrix} 22 & 15 \\ 25 & 15 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 20 & 15 \\ 22 & 15 \end{bmatrix} \mod 26
\]

\[
\begin{bmatrix} 20 & 15 \\ 22 & 15 \end{bmatrix} \times \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 2 & 5 \\ 18 & 15 \end{bmatrix} \mod 26
\]

\[
\begin{bmatrix} 2 & 5 \\ 18 & 15 \end{bmatrix} \times \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 18 & 12 \\ 18 & 15 \end{bmatrix} \mod 26
\]

\[
\begin{bmatrix} 18 & 12 \\ 18 & 15 \end{bmatrix} \times \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \equiv \begin{bmatrix} 18 & 12 \\ 18 & 15 \end{bmatrix} \mod 26
\]

The plaintext is "TO BE OR NOT TO BE."