1) a) For second order differential operators, the Green's function is given by:

$$G(x,x') = b_1 y_1(x) + b_2 y_2(x) - \frac{y_1(x)y_2(x') - y_2(x)y_1(x')}{f_0(x) W(x)} \quad x > x'$$

$$G(x,x') = b_1 y_1(x) + b_2 y_2(x) \quad x < x'$$

where $y_1(x)$ and $y_2(x)$ are the homogeneous solutions of the differential equation, $f_0(x)$ is the coefficient of the second order derivative, $W(x)$ is the Wronskian given by $W(x) = \{y_1(x)y_2(x') - y_2(x)y_1(x')\}$ and $b_1$ and $b_2$ are constants that are to be determined by boundary conditions. I have shown the derivation of this form for the $n$th order case in A-5 so this is just the special case of that derivation for 2nd order.
our operator is: \( \frac{d^2}{dx^2} + 1 \)

so \( \left[ \frac{d^2}{dx^2} + 1 \right] y(x) = 0 \)

\( \Rightarrow \)

\( y_1(x) = \cos x \quad y_2(x) = \sin x \), \( W(x) = 1 \), \( f_0(x) = 1 \)

\( G(x,x) = y_1(x) = \cos x + \frac{\cos x \sin x' - \sin x \cos x'}{1} \)

\( G(x,x') = b_1 \cos x + b_2 \sin x \quad x < x' \)

our Boundary Conditions are:

i) \( y(0) = 0 \)

ii) \( y'(1) = 0 \)

using i we have: \( b_1 = 0 \)

and using ii:

\( \frac{b_2 \sin x + b_2 \cos x + \sin x \sin x' - \cos x \cos x'}{\cos(1-x)} = 0 \)

\( b_2 = -\frac{\cos(1-x)}{\cos(1-x)} \)

\( G(x,x) = \frac{\cos(1-x)}{\cos(1-x)} \sin x + \sin(x-x') \quad x > x' \)

\( G(x,x') = \frac{\cos(1-x)}{\cos(1-x)} \sin x \quad x < x' \)
b) Now our differential operator is:

\[
\left[ \frac{d}{dx} \left( x \frac{d}{dx} \right) - \frac{n^2}{x} \right]
\]

so we have:

\[
\left[ x \frac{d^2}{dx^2} + \frac{d}{dx} - \frac{n^2}{x} \right] y = 0
\]

Let's try \( y = x^m \)

\[
y = mx^{m-1}
\]

\[
b'' = m(m-1)x^{m-2}
\]

\[
m(n-1) + m - n = 0
\]

\[
m^2 - n^2 = 0
\]

\[
m = n, m = -n
\]

Then \( G.F \) is given by:

\[
W(x) = \frac{2n}{x}, f_0(x) = x
\]

\[
G(x, x') = b_1 x^n + b_2 x^{-n} - \frac{nx^n - nx'^n}{2n} \quad x > x'
\]

\[
G(x, x') = b_1 x^n + b_2 x^{-n} \quad x < x'
\]

Our B.C.'s are:

i) \( y(a) = x_i \) ii) \( y(a) = 0 \)

Using i) \( b_2 = 0 \)

Using ii) \( b_1 = \frac{a x^n - a x'^n}{2n} \)

\[
G(x, x') = \frac{x^n - 2n}{2n} x - \frac{n x^n - n x'^n}{2n} \quad x > x'
\]

\[
G(x, x') = \frac{x^n - 2n}{2n} x^p
\]

\[
G(x, x') = \frac{x^n - 2n}{2n} x^p \quad x < x'
\]
2) \[ \frac{1}{P} \frac{\partial}{\partial P} \left( P \frac{\partial}{\partial P} \right) + \frac{1}{\rho^2} \frac{\partial^2 \phi}{\partial \rho^2} + \kappa^2 \] is our diff. op.

if we can find the eigenvalues and eigenfunctions of that operator, we can expand the Green's function in terms of these eigenfunctions

\[ \hat{\mathbf{L}} \mathbf{q}_n = \lambda n \mathbf{q}_n \implies [\hat{\mathbf{L}} - \lambda n \mathbf{L}] \mathbf{q}_n = 0 \]

we can redefine \[ k_n = \kappa - \lambda n \]

Let's use separation of variables and define:

\[ \mathbf{q}_n = P(\rho) \Phi(\rho) \] so we have:

\[ \Phi(\rho) \frac{1}{\rho} \frac{d}{d\rho} \left( \rho \frac{dP}{d\rho} \right) + \frac{P}{\rho^2} \frac{d^2 \Phi}{d\rho^2} + \kappa^2 \rho \Phi = 0 \] divide by \( \rho \)

\[ \frac{1}{\rho} \frac{d}{d\rho} \left[ \frac{d^2 \phi}{d\rho^2} + \frac{1}{\rho} \frac{d \phi}{d\rho} \right] + \frac{1}{\rho^2} \frac{1}{\rho} \frac{d^2 \phi}{d\rho^2} + \kappa^2 = 0 \]

\[ -m^2 \] where \( m \) is an integer

because at the boundary condition for \( \Phi \)

\[ \frac{d^2 \phi}{d\theta^2} = -m^2 \phi \Rightarrow \phi = A e^{im\theta} + B e^{-im\theta} \] the remaining part is:

\[ \frac{1}{\rho} \left[ \frac{d^2 \phi}{d\rho^2} + \frac{1}{\rho} \frac{d \phi}{d\rho} \right] - \frac{m^2}{\rho^2} + \kappa^2 = 0 \] multiplying by \( \rho \)

and \( \rho^2 \) and making the transformation

\[ \mathbf{f}_n = \rho \mathbf{q}_n \]
\[ \frac{\partial^2 P}{\partial \Sigma^2} + \Sigma \frac{\partial P}{\partial \Sigma} + \left[ \Sigma^2 - \nu^2 \right] P = 0 \]

This is the Bessel Differential eq. whose solutions are Bessel fn.'s.

\[ P(\rho) = C_m J_\nu(k_m \rho) + D_m Y_\nu(k_m \rho) \]

so now we can expand our G.F using \( \alpha_\nu \) in \( P(\rho) \) since \( \text{Dmn} = 0 \) since we want \( P(\rho) \) to be finite at the origin.

\[ G(p, \phi) = \sum_m \sum_n F_{mn} J_\nu(k_{mn} \rho) [e^{-i\phi m} - e^{-i\phi n}] \]

where \( F_{mn} \)'s are constants to be determined by the zeros of the Bessel fn.
3) $L q + q(x) y + \lambda y = f(x)$

$$L = \frac{d}{dx} \left( \frac{d}{dx} \lambda \left( \frac{d}{dx} \right) \right)$$

We are given that $\left[ L + q(x) \right] \psi_n(x) = \lambda_n \psi_n(x)$

G.F. can be expanded in terms of any complete basis (actually it is true for all functions)

$$G(x, \xi) = \sum_n \psi_n(x) \psi_n(\xi)$$

Now let's apply $\left[ L + q(x) \right]$ to both sides

$$\left[ L + q(x) \right] G(x, \xi) = \sum_n \psi_n(x) \left[ \lambda + \lambda_n \right] \psi_n(\xi)$$

since the G.F. satisfies

$$\frac{d}{dx} G(x, x) = \delta(x-x)$$

of the operator $\frac{d}{dx}$

Finally using the orthogonality of $\psi_n$'s we can find $\lambda_n$:

$$\psi_n(\xi) = \frac{\frac{d}{dx} \psi_n(x)}{\lambda + \lambda_n}$$

so G.F. is given by:

$$G(x, \xi) = \sum_n \frac{\psi_n(\xi) \psi_n(x)}{\lambda + \lambda_n}$$
4) Let's first start with \( n=2 \), we have:

\[
\frac{f_0(x)}{\text{d}x^2} + f_1(x) \frac{\text{d}}{\text{d}x} + f_2(x)
\]

\( \tilde{L} \)

Let \( u_1 \) and \( u_2 \) be the solutions of this equation:

\[
\tilde{L} u_1 = 0 \\
\tilde{L} u_2 = 0
\]

\[
\left[ f_0(x) \frac{\text{d}^2}{\text{d}x^2} + f_1(x) \frac{\text{d}}{\text{d}x} + f_2(x) \right] u_1 = 0 \\
\left[ f_0(x) \frac{\text{d}^2}{\text{d}x^2} + f_1(x) \frac{\text{d}}{\text{d}x} + f_2(x) \right] u_2 = 0
\]

Multiply by \( u_2 \) from the left

\[
\left[ f_0(x) \frac{\text{d}^2}{\text{d}x^2} + f_1(x) \frac{\text{d}}{\text{d}x} + f_2(x) \right] u_2 \cdot u_1 = 0
\]

\[
\left[ f_0(x) \frac{\text{d}^2}{\text{d}x^2} + f_1(x) \frac{\text{d}}{\text{d}x} + f_2(x) \right] u_2 \cdot u_1 = 0
\]

Multiply by \( u_2 \) from the left

\[
\left[ f_0(x) \frac{\text{d}^2}{\text{d}x^2} + f_1(x) \frac{\text{d}}{\text{d}x} + f_2(x) \right] u_2 \cdot u_1 = 0
\]

and subtract

\[
\left[ f_0(x) \left[ u_2 u_1 - u_1 u_2 \right] + f_1(x) \left[ u_2 u_1 - u_1 u_2 \right] \right] = 0
\]

\[
W(x) = \frac{f_1(x)}{f_2(x)} W(x)
\]

\[
W(x) = - \frac{f_1(x)}{f_2(x)} W(x)
\]

Now let's do the same thing for a general \( n \).

For a general \( n \) we can write the Wronskian as:

\[
W(x) = \sum_{i=1}^{n} \left( n_i \right) W_i(x)
\]

Here \( \{i\} \) is the antisymmetric tensor; it gives zero when its two repeated indices.
For the general case let us assume $u_1, \ldots, u_n$ are the solutions of the diff eq. $L u_i = 0$ for $i = 1, \ldots, n$. We can write a similar eq. to the one we have written for the $n=2$ case but then we may have many eq's but we can write them in a closed form using the antisymmetric Levi-Civita tensor.

\[ \sum_{i_1, \ldots, i_n} \varepsilon_{i_1 \ldots i_n} u_{i_1} u_{i_2} \cdots u_{i_n} \left( f_0(x) \frac{df}{dx} + \cdots + f_n(x) \right) u_i = 0 \]

\[ \sum_{i_1, \ldots, i_n} \varepsilon_{i_1 \ldots i_n} \left( u_{i_1} u_{i_2} \cdots u_{i_n} f_0(x) + \left( u_{i_1} u_{i_2} \cdots u_{i_n} \right) f_1(x) \right) W(x) = 0 \]

because all the remaining terms will contain two same order derivative terms and since $\varepsilon$ is an antisymmetric tensor and we are summing over $i_1, i_2, \ldots$ they will sum up to zero so we have

\[ W(x) = - \frac{f_0(x)}{f_n(x)} W(x) \]
5) \( \frac{d^n}{dx^n} + f_1(x) \frac{d^{n-1}}{dx^{n-1}} + \cdots + f_n(x) \)

We can write GF as

\[
G = a_1 u_1(x) + \cdots + a_n u_n(x)
\]

\[
G = b_1 u_1(x) + \cdots + b_n u_n(x)
\]

taking \( n-1 \) integrals and using the cent. of GF we have:

\[
a_1 u_1(x) + \cdots + a_n u_n(x) = b_1 u_1(x) + \cdots + b_n u_n(x)
\]

\[
\begin{align*}
(a_1, u_1(x)) &+ \cdots + (a_n, u_n(x)) = b_1 u_1(x) + \cdots + b_n u_n(x) \\
&- \left( b_1 u_1(x) + \cdots + b_n u_n(x) \right) - \left( b_1 u_1(x) + \cdots + b_n u_n(x) \right) = 1 = f(x)
\end{align*}
\]

we can write these set of equations in matrix form.

\[
\begin{bmatrix}
 x(1) \\
 x(2) \\
 \vdots \\
 x(n)
\end{bmatrix} + \begin{bmatrix}
 b_1 \\
 b_2 \\
 \vdots \\
 b_n
\end{bmatrix} = \begin{bmatrix}
 a_1 \\
 a_2 \\
 \vdots \\
 a_n
\end{bmatrix}
\]

\[
(a_1, b_1) = \begin{bmatrix}
 b_1 \\
 b_2 \\
 \vdots \\
 b_n
\end{bmatrix} \frac{1}{det(M)} \mathrm{remains \ as \ matrix}
\]

where \( \det(M) \)
So we can now write the GF as

\[ G = \sum_{i} b_i \int \frac{1}{W(x)} \Phi(x) \, dW(x) \]

\[ = \sum_{i} b_i \left[ U_i(\phi) \right] + \frac{1}{W(x)} \left[ U_i(\phi) \right] \left[ \begin{array}{c} a \frac{u_2 - u_1}{u_1 - u_2} \\ \vdots \\ \frac{1}{u_1 - u_2} \\ \vdots \\ \frac{1}{u_1 - u_N} \end{array} \right] + U_2(x) \left[ \begin{array}{c} u_1 \\ \vdots \\ u_N \end{array} \right] \]

\[ + \cdots + U_N(x) \left[ \begin{array}{c} u_1 \\ \vdots \\ u_N \end{array} \right] \]

\[ G = \sum_{i} b_i \Phi_i(x) + \frac{N(x, x)}{F_0(x) W(x)} \Theta(x-x) \]