Solution 7.1

The general solutions for barrier potential are;

\[
\Psi(x) = \begin{cases} 
A e^{ikx} + Be^{-ikx} & \text{if } x < 0 \\
Ce^{-\alpha x} + De^{\alpha x} & \text{if } 0 < x < L \\
F e^{ikx} & \text{if } x > L 
\end{cases}
\]

where;

\[
\alpha = \sqrt{\frac{2m}{\hbar^2}(V_0 - E)} \quad k = \sqrt{\frac{2mE}{\hbar^2}}
\]

The wavefunction and its first derivative should be continuous at the boundaries, we get;

\[
\begin{align*}
A + B &= C + D \\
\alpha(D - C) &= \frac{\alpha(D - C)}{ik}(A - B) \\
F e^{iKL} &= Ce^{-\alpha L} + De^{\alpha L} \\
\alpha(De^{\alpha L} - Ce^{-\alpha L}) &= \frac{i}{k}F e^{iKL} \\
\end{align*}
\]

We have 5 unknowns (A,B,C,D and F) and 4 equations (Eqs.5-8). The last unknown can be determined from normalization condition. But its unnecessary in order to calculate the transmission coefficient Eq.9;

\[
T = \frac{\nu_1 F^* F}{\nu_1 A^* A} = \frac{|F|^2}{|A|^2}
\]

All we need to find is the relation between F and A. Let me first eliminate B from Eqs.5-6. Multiply Eq.5 with \(ik\) and add with Eq.6, we get;

\[
2ikA = D(\alpha + ik) + C(ik - \alpha)
\]

Multiply Eq.7 with \(-ik\) and add with Eq.8, we get;

\[
0 = De^{\alpha L}(-ik + \alpha) + Ce^{-\alpha L}(-ik - \alpha)
\]
Again multiply Eq.7 with $\alpha$ and add with Eq.8, we get;

$$F e^{ikL}(ik + \alpha) = 2\alpha De^{\alpha L} \tag{12}$$

Now eliminate $C$ from Eq.10 and Eq.11;

$$2i k A = D(\alpha + i k) + \frac{(ik - \alpha) De^{\alpha L}(\alpha - ik)}{(\alpha + ik)e^{-\alpha L}} \tag{13}$$

Then using Eq.12 and Eq.13 the relation between $A$ and $F$ is;

$$\frac{F e^{ikL}(ik + \alpha)}{2ikA} = \frac{2\alpha e^{\alpha L}}{(\alpha + ik) + \frac{(ik - \alpha)e^{\alpha L}(\alpha - ik)}{(\alpha + ik)e^{-\alpha L}}} \tag{14}$$

or;

$$\frac{F}{A} = \frac{4ik\alpha}{e^{ikL}[(\alpha + ik)^2e^{-\alpha L} + (ik - \alpha)e^{\alpha L}(\alpha - ik)]} \tag{15}$$

Take absolute value and then the square of both sides;

$$T = \frac{|F|^2}{|A|^2} = \frac{16k^2\alpha^2}{(\alpha + ik)^2e^{-\alpha L} + (ik - \alpha)e^{\alpha L}(\alpha - ik)^2} \tag{16}$$

$$T = \frac{16k^2\alpha^2}{e^{-\alpha L}(\alpha^2 - k^2 + 2ik\alpha) + e^{\alpha L}(\alpha^2 + k^2 + 2ik\alpha)} \tag{17}$$

$$T = \frac{16k^2\alpha^2}{|e^{-\alpha L}(\alpha^2 - k^2) + e^{\alpha L}(\alpha^2 + k^2) + 2ik\alpha(e^{-\alpha L} + e^{\alpha L})|^2} \tag{18}$$

$$T = \frac{16k^2\alpha^2}{[e^{-\alpha L}(\alpha^2 - k^2) + e^{\alpha L}(\alpha^2 + k^2)]^2 + [2k\alpha(e^{-\alpha L} + e^{\alpha L})]^2} \tag{19}$$

$$T = \frac{16k^2\alpha^2}{[(\alpha^4 + k^4 - 2\alpha^2k^2)(e^{-2\alpha L} + e^{2\alpha L} - 2) + 4k^2\alpha^2(e^{-2\alpha L} + e^{2\alpha L} + 2)]} \tag{20}$$

$$T = \frac{16k^2\alpha^2}{[(e^{-2\alpha L} + e^{2\alpha L})(\alpha^4 + k^4 + 2\alpha^2k^2) - 2(\alpha^4 - k^4 + 12k^2\alpha^2)]} \tag{21}$$

$$T = \frac{(e^{-2\alpha L} + e^{2\alpha L})(\alpha^2 + k^2)^2 - 2(\alpha^2 + k^2)^2 + 16k^2\alpha^2}{16k^2\alpha^2} \tag{22}$$
\[ T = \left( \frac{e^{-2\alpha L} + e^{2\alpha L} - 2(\alpha^2 + k^2)^2}{16k^2\alpha^2} \right) + 1 \]^{-1} \quad (23) 

\[ T = \left( \frac{(e^{-\alpha L} - e^{\alpha L})^2(\alpha^2 + k^2)^2}{16k^2\alpha^2} \right) + 1 \]^{-1} \quad (24)

Now it looks so simple, insert \( \alpha \) and \( k \) from Eq.4;

\[ T = \left( \frac{(e^{-\alpha L} - e^{\alpha L})^2 \left( \frac{2m(V_0 - E)}{\hbar^2} \right) + 2mE}{16 \frac{2mE}{\hbar^2} \frac{2m(V_0 - E)}{\hbar^2}} \right) + 1 \]^{-1} \quad (25)

\[ T = \left( \frac{(e^{-\alpha L} - e^{\alpha L})^2 \left( \frac{2mV_0}{\hbar^2} \right)^2}{16 \frac{2mE}{\hbar^2} \frac{2m(V_0 - E)}{\hbar^2}} \right) + 1 \]^{-1} \quad (26)

\[ T = \left[ \frac{(e^{-\alpha L} - e^{\alpha L})^2}{16 \frac{E V_0}{\hbar^2} (1 - \frac{E}{V_0})} \right]^{-1} = \left[ 1 + \frac{\sinh^2 \alpha L}{4 \frac{E}{V_0} (1 - \frac{E}{V_0})} \right]^{-1} \quad (27) \]

Solution 7.2

\[ U(x) = -\alpha[\delta(x + a) + \delta(x - a)] \] (28)

In the region where the potential is zero, the solutions of the wave equation are in the form \( e^{\pm \kappa x} \), but the solutions should vanish at \( x = \pm \infty \) so that

\[ \Psi(x) = Ae^{\kappa x} \quad x < -a \] (29)

\[ \Psi(x) = Be^{\kappa x} + Ce^{-\kappa x} \quad -a < x < a \] (30)

\[ \Psi(x) = De^{-\kappa x} \quad x > a \] (31)

where \( \kappa \) is for \( E < 0 \);

\[ \kappa = \sqrt{-2mE k^2} \] (32)

The continuity of wavefunction at \( x = \pm a \) yields;

\[ Ae^{-\kappa a} = Be^{-\kappa a} + Ce^{\kappa a} \] (33)

\[ Be^{\kappa a} + Ce^{-\kappa a} = De^{-\kappa a} \] (34)

The first derivative of wavefunction at \( x = \pm a \) is discontinuous.
According to jump condition;
\[ B\kappa e^{-\kappa a} - C\kappa e^{-\kappa a} - A\kappa e^{-\kappa a} = -\frac{2m}{\hbar^2} \alpha A e^{-\kappa a} \] (35)
\[ -D\kappa e^{-\kappa a} - B\kappa e^{-\kappa a} + C\kappa e^{-\kappa a} = -\frac{2m}{\hbar^2} \alpha D e^{-\kappa a} \] (36)
or;
\[ B\kappa e^{-\kappa a} - C\kappa e^{-\kappa a} = A e^{-\kappa a} [-\frac{2m}{\hbar^2} \alpha + \kappa] \] (37)
\[ -B\kappa e^{-\kappa a} + C\kappa e^{-\kappa a} = D e^{-\kappa a} [-\frac{2m}{\hbar^2} \alpha + \kappa] \] (38)
To determine the allowed bound states we need to solve Eqs. 33-34 and Eqs.37-38. Eliminate \( A \) from Eq.33 and Eq.37:
\[ \kappa Be^{-\kappa a} - \kappa Ce^{-\kappa a} = [-\frac{2m}{\hbar^2} \alpha + \kappa] [Be^{-\kappa a} + Ce^{-\kappa a}] \]
\[ \frac{2m}{\hbar^2} \alpha Be^{-\kappa a} = Ce^{-\kappa a} [2\kappa - \frac{2m}{\hbar^2} \alpha] \] (39)
Similarly eliminate \( D \) from Eq.34 and Eq.38:
\[ -\kappa Be^{\kappa a} + \kappa Ce^{-\kappa a} = [-\frac{2m}{\hbar^2} \alpha + \kappa] [Be^{\kappa a} + Ce^{-\kappa a}] \]
\[ \frac{2m}{\hbar^2} \alpha Ce^{-\kappa a} = Be^{\kappa a} [2\kappa - \frac{2m}{\hbar^2} \alpha] \] (40)
Divide Eq.39 to Eq.40:
\[ \frac{B}{C} = \frac{C}{B} \Rightarrow B = \pm C \] (41)
It is easy to see that from Eq.30 and Eq.41 that if \( B=C \) the solution is even (cosh \( \kappa x \)) and if \( B=-C \) the solution is odd (sinh \( \kappa x \)). For the even solution using Eq.39 (or Eq.40);
\[ \frac{2m}{\hbar^2} \alpha e^{-\kappa a} = e^{\kappa a} [2\kappa - \frac{2m}{\hbar^2} \alpha] \] (42)
or;

\[ e^{-2\kappa a} = \frac{\hbar^2 \kappa}{m\alpha} - 1 \]  \hspace{1cm} (43)

If you plot Eq.43 you see that for the even solution there is always a single bound state. That means \( e^{-2\kappa a} \) and \( \frac{\hbar^2 \kappa}{m\alpha} - 1 \) curves always intersect with each other at the region \( \kappa > 0 \). For instance let \( \alpha = \hbar^2 / ma \) the intersection point is located at \( \kappa a \approx 1.1 \).

For the odd solution the situation different. The existence of the bound state depends on the strength of the potential, \( \alpha \). According to Eq.40 and \( B=-C \) we get

\[ e^{-2\kappa a} = -\left( \frac{\hbar^2 \kappa}{m\alpha} - 1 \right) \]  \hspace{1cm} (44)

If you plot \( e^{-2\kappa a} \) and \( -\left( \frac{\hbar^2 \kappa}{m\alpha} - 1 \right) \), the two curves intersect or do not intersect depending on \( \alpha \) at the region \( \kappa > 0 \) (obviously \( \kappa = 0 \) is another intersection point, and also another root of Eq.44, but this is not an acceptable solution because you cannot normalize the wavefunction). There will only be an intersection point if the slope of the \( e^{-2\kappa a} \) is smaller than the slope of \( -\left( \frac{\hbar^2 \kappa}{m\alpha} - 1 \right) \) at the origin, that is;

\[ -2a < -\frac{\hbar^2}{m\alpha} \Rightarrow \alpha > \frac{\hbar^2}{2ma} \]  \hspace{1cm} (45)

Finally I can conclude that the total number of bound states can be one or two depending on \( \alpha \).

Using Eq.32 the energies can be found from;

\[ E = -\frac{\hbar^2 \kappa^2}{2m} \]  \hspace{1cm} (46)

For \( \alpha = \hbar^2 / ma \), and for the even solution;

\[ E = -\frac{\hbar^2 \kappa^2}{2m} = -\frac{\hbar^2 \kappa^2 a^2}{2ma^2} \approx -(1.1)^2 \frac{\hbar^2}{2ma^2} \]  \hspace{1cm} (47)

And for the odd solution (for \( \alpha = \hbar^2 / ma \) the root is located at
\( \kappa a \approx 0.8 \);

\[
E = -\frac{\hbar^2 \kappa^2}{2m} = -\frac{\hbar^2 \kappa^2 a^2}{2ma^2} \approx -(0.8)^2 \frac{\hbar^2}{2ma^2}
\]  

(48)

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