1) Find the radius and the interval of convergence of the series

$$\sum_{n=2}^{\infty} \frac{(-2)^n(n+1)}{(3n^2+1)\ln(n)}(x-5)^n$$

**Solution:** We have

$$\lim_{n \to \infty} \frac{|(-2)^{n+1}(n+2)(x-5)^{n+1}|}{|(-2)^n(n+1)(x-5)^n|} = \lim_{n \to \infty} \frac{2(n+2)(3n^2+1)\ln(n)|x-5|}{(n+1)(3(n+1)^2+1)\ln(n+1)} = 2|x-5|$$

Hence by Ratio Test, $$\sum_{n=2}^{\infty} \frac{|(-2)^n(n+1)|}{(3n^2+1)\ln(n)}(x-5)^n$$ converges when $$2|x-5| < 1$$.

This means by Absolute Convergence Test, $$\sum_{n=2}^{\infty} \frac{(-2)^n(n+1)}{(3n^2+1)\ln(n)}(x-5)^n$$ converges when $$x$$ is in the interval $$\left(\frac{9}{2}, \frac{11}{2}\right)$$. Now let us consider left end point of this interval.

In case $$x = \frac{9}{2}$$, the series $$\sum_{n=2}^{\infty} \frac{(-2)^n(n+1)}{(3n^2+1)\ln(n)}(x-5)^n = \sum_{n=2}^{\infty} \frac{(n+1)}{(3n^2+1)\ln(n)}$$ diverges because $$\lim_{n \to \infty} \frac{(n+1)}{(3n^2+1)\ln(n)} = 1$$ and $$\sum_{n=2}^{\infty} \frac{1}{n \ln(n)}$$ diverges by integral test as

$$\lim_{r \to \infty} \int_{2}^{r} \frac{1}{x \ln(x)} \, dx = \lim_{r \to \infty} \ln(\ln(r)) - \ln(\ln(2)) = \infty.$$

This also means for $$|x-5| > \frac{1}{2}$$, we have $$\sum_{n=2}^{\infty} \frac{(-2)^n(n+1)}{(3n^2+1)\ln(n)}(x-5)^n$$ diverges. In case $$x = \frac{11}{2}$$, the series $$\sum_{n=2}^{\infty} \frac{(-2)^n(n+1)}{(3n^2+1)\ln(n)}(x-5)^n = \sum_{n=2}^{\infty} \frac{(-1)^n(n+1)}{(3n^2+1)\ln(n)}$$ converges by Alternating Series Test because $$\left\{ \frac{(n+1)}{(3n^2+1)\ln(n)} \right\}$$ is a decreasing sequence with positive terms which converges to 0.

So the radius of convergence of $$\sum_{n=2}^{\infty} \frac{(-2)^n(n+1)}{(3n^2+1)\ln(n)}(x-5)^n$$ is $$\frac{1}{2}$$.

The interval of convergence of $$\sum_{n=2}^{\infty} \frac{(-2)^n(n+1)}{(3n^2+1)\ln(n)}(x-5)^n$$ is $$\left(\frac{9}{2}, \frac{11}{2}\right)$$.
2) Find the absolute maximum and minimum values of the function

\[ f(x, y) = yx^2 - y^2 + 4 \]

on the unit disk \( D = \{(x, y) | x^2 + y^2 \leq 1 \} \).

**Solution:** First we will try to find the critical points in the interior of the unit disk.

\[
0 = f_x = 2xy \\
0 = f_y = x^2 - 2y \
\iff \quad x = 0 \text{ or } y = 0 \quad x^2 = 2y \quad \iff \quad x = 0 \text{ and } y = 0
\]

Only critical point in the interior of \( D \) is \((0, 0)\). On the boundary of \( D \) we apply Lagrange multipliers.

\[
f (x, y) = yx^2 - y^2 + 4 \\
g (x, y) = x^2 + y^2 - 1
\]

We must have \( \nabla f = \lambda \nabla g \). This yields

\[
2xyi + (x^2 - 2y)j = \lambda (2xi + 2yj) \quad \Rightarrow \quad 2xy = \lambda 2x \quad \Rightarrow \quad x = 0 \text{ or } y = \lambda \\
x^2 - 2y = \lambda 2y
\]

Thus \( x = 0 \) or \( y = \lambda \).

**Case 1.** \( y = \lambda \): We get \(-2y = \lambda 2y \) which yields \( y = 0 \) or \( \lambda = -1 \). \( x = 0 \), \( y = 0 \) on the boundary is impossible. So we must have \( \lambda = -1 \). In this case we must have

\[
x^2 + y^2 = 1 \quad \Rightarrow \quad y^2 = 1 \quad \Rightarrow \quad y = \pm 1
\]

since \( (x, y) \) is on the boundary. Thich yields the points \((0, -1)\) and \((0, 1)\) for possible extremum.

**Case 2.** \( x = 0 \): We get \(-2y = \lambda 2y \) which yields \( y = 0 \) or \( \lambda = -1 \). \( x = 0 \), \( y = 0 \) on the boundary is impossible. So we must have \( \lambda = -1 \). In this case we must have

\[
x^2 + y^2 = 1 \quad \Rightarrow \quad y^2 = 1 \quad \Rightarrow \quad y = \pm 1
\]

which yields the point \((0, -1)\) for possible extremum. If \( \lambda = \frac{1}{3} \), then

\[
x^2 = 2\lambda + 2\lambda^2 = \frac{8}{9} \quad \text{and} \quad y = \frac{1}{3}
\]

which yields the points \( \left(\frac{2\sqrt{2}}{3}, \frac{1}{3}\right) \) and \( \left(-\frac{2\sqrt{2}}{3}, \frac{1}{3}\right) \) for possible extremum. We now evaluate the value of \( f \) for each of the points:

\[
f (0, 0) = 4, \quad f (0, 1) = f (0, -1) = 3, \quad f \left(\frac{2\sqrt{2}}{3}, \frac{1}{3}\right) = f \left(-\frac{2\sqrt{2}}{3}, \frac{1}{3}\right) = \frac{113}{27} > 4
\]

Thus absolute maximum of \( f \) is \( \frac{113}{27} \) attained at the points \( \left(\frac{2\sqrt{2}}{3}, \frac{1}{3}\right) \) and \( \left(-\frac{2\sqrt{2}}{3}, \frac{1}{3}\right) \), and the absolute minimum of \( f \) is \( 3 \) attained at the points \((0, 1)\) and \((0, -1)\).
3) Find the work done by the vector field 
\[ \mathbf{F}(x, y, z) = (2xz + 2)i + (2y + z)j + (x^2 + y)k \]
over the curve \( \mathbf{r}(t) = (\sin(t) + 2 \cos(t) - 2)i + (e^{t^2 - \pi t})j + (\frac{t}{\pi} - 5)k \) for \( t \) in \([0, \pi]\).

**Solution:** Define \( f(x, y, z) = x^2z + 2x + y^2 + zy \). Then notice that \( \nabla f = \mathbf{F} \). Hence we have
\[
\int_{t=0}^{t=\pi} \mathbf{F} \, d\mathbf{r} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = -75 - (-4) = -71
\]

4) Integrate \( f(x, y, z) = x^2 + y + 3z \) over the line segment joining \((0, 1, 2)\) to \((-1, -1, -2)\).

**Solution:** A parametrization for the line segment can be given as follows:
\[ \mathbf{r}(t) = (-t)i + (1 - 2t)j + (2 - 4t)k \] for \( 0 \leq t \leq 1 \)

For \( 0 \leq t \leq 1 \) we have
\[ f(\mathbf{r}(t)) = (-t)^2 + (1 - 2t) + 3(2 - 4t) = t^2 - 14t + 7 \]
and
\[ |\mathbf{v}(t)| = \left| \frac{d\mathbf{r}}{dt} \right| = |-i - 2j - 4k| = \sqrt{(-1)^2 + (-2)^2 + (-4)^2} = \sqrt{21} \]
Hence we have
\[
\int_{t=0}^{t=1} f \, ds = \int_{0}^{1} f(\mathbf{r}(t)) |\mathbf{v}(t)| \, dt = \int_{0}^{1} (t^2 - 14t + 7)\sqrt{21} \, dt = \frac{\sqrt{21}}{3}
\]
5) Let \( C \) be a curve that encloses a region \( R \) such that the area of the region \( R \) is \( 10\pi \) and the interior of the region \( R \) contains the unit disk \( D = \{(x, y) \mid x^2 + y^2 \leq 1\} \). Compute the integral

\[
\oint_C \left( \frac{x - 2y}{x^2 + y^2} \right) dx + \left( \frac{2x + y}{x^2 + y^2} + 3x \right) dy
\]

where the integral is over \( C \) once in the counterclockwise direction.

**Solution:** Let

\[
F = \frac{x - 2y}{x^2 + y^2} \mathbf{i} + \left( \frac{2x + y}{x^2 + y^2} + 3x \right) \mathbf{j}
\]

Then

\[
\oint_C F \, dr = \oint_C \frac{x - 2y}{x^2 + y^2} \, dx + \left( \frac{2x + y}{x^2 + y^2} + 3x \right) \, dy
\]

We have

\[
k \cdot (\nabla \times F) = \frac{\partial}{\partial x} \left( \frac{2x + y}{x^2 + y^2} + 3x \right) - \frac{\partial}{\partial y} \left( \frac{x - 2y}{x^2 + y^2} \right)
\]

\[
= -2 \frac{(x^2 + xy - y^2)}{(x^2 + y^2)^2} + 3 + 2 \frac{(x^2 + xy - y^2)}{(x^2 + y^2)^2} = 3
\]

Thus

\[
\iint_{R-D} k \cdot (\nabla \times F) \, dA = \iint_{R-D} 3 \, dA = 3 \text{ (area of } R - D) =
\]

\[
= 3 \left( (\text{area of } R) - (\text{area of } D) \right) = 3 \left( 10\pi - \pi \right) = 27\pi
\]

\( \partial D \) is the circle with radius 1, hence it can be parametrized as \( x = \cos t, y = \sin t \) where \( 0 \leq t \leq 2\pi \). Thus have

\[
\int_{\partial D} F \, dr = \int_0^{2\pi} (\cos t - 2 \sin t) (-\sin t) \, dt + (2 \cos t + \sin t + 3 \cos t) (\cos t) \, dt
\]

\[
= \int_0^{2\pi} (2 \sin^2 t - \cos t \sin t + 5 \cos^2 t + \sin t \cos t) \, dt
\]

\[
= \int_0^{2\pi} (2 + 3 \cos^2 t) \, dt = 7\pi
\]

By Green’s Theorem

\[
\iint_C F \, dr - \int_{\partial D} F \, dr = \iint_{R-D} k \cdot (\nabla \times F) \, dA
\]

We thus have

\[
\oint_C F \, dr = 7\pi + 27\pi = 34\pi
\]
6) Evaluate \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( C \) is the intersection of \( z = x^2 + y^2 + 1 \) and \( z = 2y + 1 \) oriented clockwise as viewed from above and \( \mathbf{F} = (\sin(x^2), y^3, z \ln(z) - x) \)

**Solution:**

We know that

\[
\oint_C \mathbf{F} d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma
\]

We calculate \( \nabla \times \mathbf{F} \) and \( \mathbf{n} \).

\[
\begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
\sin(x^2) & y^3 & (z \ln z - x)
\end{vmatrix}
= (0 - 0) \mathbf{i} - (0 - 0) \mathbf{j} + (0 - 0) \mathbf{k} = \mathbf{j}
\]

Let us find the intersection of \( z = x^2 + y^2 + 1 \) and \( z = 2y + 1 \).

\[
x^2 + y^2 + 1 = 2y + 1 \implies x^2 + y^2 - 2y + 1 = 1 \implies x^2 + (y - 1)^2 = 1
\]

Thus if we let

\[
R = \{ (x, y) : x^2 + (y - 1)^2 \leq 1 \}
\]

then the surface integral can be evaluated as follows.

\[
\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma = \iint_R \mathbf{j} \cdot \frac{2\mathbf{j} - \mathbf{k}}{\sqrt{5}} \sqrt{5} dA
\]

\[
= \iint_R 2 dA = 2 \text{ (area of } R) = 2\pi
\]

This shows

\[
\int_C \mathbf{F} d\mathbf{r} = 2\pi
\]
7) Let $Q$ be the solid bounded by the paraboloid $z = 4 - x^2 - y^2$ and the $xy$-plane. Find the outward flux of the vector field $\mathbf{F} = \langle x^3, y^3, z \rangle$ over the boundary of $Q$.

**Solution:**

By Divergence Theorem

$$\text{Flux} = \iiint_{\partial Q} \mathbf{F} \cdot \mathbf{n} \, d\sigma = \iiint_{Q} \nabla \cdot \mathbf{F} \, dV$$

Thus

$$\text{Flux} = \iiint_{Q} \nabla \cdot \mathbf{F} \, dV$$

$$= \iiint_{Q} (3x^2 + 3y^2 + 1) \, dV$$

$$= \int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{\sqrt{4-r^2}} (3r^2 + 1) \, r \, dz \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{2} \left( 4 - r^2 \right) (3r^2 + 1) \, r \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \int_{0}^{2} (-3r^5 + 11r^3 + 4r) \, dr \, d\theta$$

$$= \int_{0}^{2\pi} \left( -\frac{1}{2}r^6 + \frac{11}{4}r^4 + 2r^2 \right) \bigg|_{0}^{2} \, d\theta$$

$$= \int_{0}^{2\pi} 20 \, d\theta = 40\pi$$