1a) \( f \) is continuous at 0 if
\[
\lim_{x \to 0} f(x) = f(0) = 0.
\]
\[
\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{(\cos x - 1)(\cos x + 1)}{x(\cos x + 1)} = \lim_{x \to 0} \frac{\cos^2 x - 1}{x(\cos x + 1)}
\]
\[
= \lim_{x \to 0} \frac{-\sin^2 x}{x(1 + \cos x)} = \lim_{x \to 0} \frac{\sin x - \sin x}{x} = 1, \quad \frac{0}{1 + 1} = 0.
\]
Hence, \( f \) is continuous at 0.

1b) From the definition, we have
\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\cos x - 1}{x} = \lim_{x \to 0} \frac{\cos x - 1}{x^2}
\]
\[
= \lim_{x \to 0} \frac{-\sin^2 x}{x^2(1 + \cos x)} \quad \text{(by arguing similarly as we did above)}
\]
\[
= \lim_{x \to 0} \frac{\sin x \sin x}{x} \frac{-1}{1 + \cos x} = 1, \quad \frac{-1}{1 + 1} = -\frac{1}{2}.
\]
Hence, \( f \) is differentiable at 0 and \( f'(0) = -\frac{1}{2} \).

2a) \( g \) is a continuous function on \([0, \pi/2]\).
\[
g(0) = 0 - 1 = -1 \quad \text{and} \quad g(\pi/2) = \pi^2/4 - 0 = \pi^2/4.
\]
Since \(-1/2\) is between \( g(0) = -1 \) and \( g(\pi/2) = \pi^2/4 \), by the Intermediate Value Theorem, there is a number \( c \in (0, \pi/2) \) such that \( g(c) = -1/2 \).

2b) We have \( p(-2) = -1, \ p(-1) = 3, \ p(0) = 1, \ p(1) = -1 \) and \( p(2) = 3 \). Since \( p(x) \) is continuous on \([-2, -1]\) and 0 is between \( p(-2) = -1 \) and \( p(-1) = 3 \), by the Intermediate Value Theorem, there is a \( c_1 \in (-2, -1) \) such that \( p(c_1) = 0 \). Arguing similarly we find that there is \( c_2 \in (0, 1) \) and \( c_3 \in (1, 2) \) such that \( p(c_2) = p(c_3) = 0 \). Since a third degree polynomial can have at most three roots, \( c_1, c_2 \) and \( c_3 \) are all the roots of \( p(x) \).

3a) If \( f \) is differentiable at 0, then \( f \) is continuous at 0. Thus, we need to have that
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x) = f(0).
\]
Using the fact that the functions \( g(x) = \cos x \) and \( h(x) = a + bx + cx^2 \) are both continuous at 0, we find that
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (a + bx + cx^2) = h(0) = a
\]
and

\[ \lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} \cos x = g(0) = f(0) = \cos(0) = 1. \]

Hence, we conclude that \( a = 1 \). If \( f \) is differentiable at 0, then from the definition we have

\[ \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0}. \]

Using the fact that the functions \( g(x) = \cos x \) and \( h(x) = 1 + bx + cx^2 \) are both differentiable at 0 and \( f(0) = g(0) = h(0) \), we find that

\[ \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{g(x) - g(0)}{x - 0} = g'(0) = -\sin(0) = 0. \]

Hence, \( b = 0 \). Arguing similarly, we find that if \( f' \) is differentiable at 0 that is \( f''(0) \) exists, then

\[ h''(0) = g''(0) \text{ which implies } 2c = -\cos(0) \text{ or } c = -1/2. \]

Next, if \( f'''(0) \) exists then \( h'''(0) = g'''(0) \) that is \( 0 = \sin(0) \) which is satisfied. Hence, we conclude that for \( a = 1, b = 0 \) and \( c = -1/2, f''(0) \) exists. \( f^{(4)}(0) \) does not exist since \( h^{(4)}(0) = g^{(4)}(0) \) that is \( 0 = \cos(0) \) is not satisfied.

3b) Let \( g(x) = 3x^2 + 2 \sin x - 1 \), then \( g(\mp 1) = 2(1 + \sin(\mp 1)) > 0 \). Clearly,

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \frac{g(x)}{x^2 - 1} = +\infty, \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{g(x)}{x^2 - 1} = -\infty \]

and

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} \frac{g(x)}{x^2 - 1} = -\infty, \quad \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{g(x)}{x^2 - 1} = +\infty. \]

Hence, \( x = 1 \) and \( x = -1 \) are vertical asymptotes of \( f \).

Since \(-1/x^2 \leq (\sin x)/x^2 \leq 1/x^2\),

\[ \lim_{x \to \infty} \frac{\sin x}{x^2} = \lim_{x \to \infty} \frac{\sin x}{x^2} = 0. \]

If \( x \neq 0 \), then

\[ f(x) = \frac{3x^2 + 2 \sin x - 1}{x^2 - 1} = \frac{3 + 2 \frac{\sin x}{x^2} - \frac{1}{x^2}}{1 - \frac{1}{x^2}}. \]

Therefore,

\[ \lim_{x \to \infty} f(x) = \lim_{x \to \infty} f(x) = \frac{3 - 2 \cdot 0 - 0}{1 - 0} = 3. \]
Hence, $y = 3$ is an horizontal asymptote of $f$.

4a) Recall that $f$ is a differentiable function on $(-5, 10)$ and 

$$f\left(\frac{5}{2}\right) = 10, \quad f\left(\frac{3}{2}\right) = 0, \quad \text{and} \quad f'\left(\frac{3}{2}\right) = 3.$$ 

Since $x - \frac{1}{2}$ is continuous at 3 and $f$ is continuous at $3 - \frac{1}{2} = \frac{5}{2}$, $f(x - \frac{1}{2})$ is continuous at 3 and so we have

$$\lim_{x \to 3} f(x - \frac{1}{2}) = f\left(3 - \frac{1}{2}\right) = f\left(\frac{5}{2}\right) = 10 \quad \text{and} \quad \lim_{x \to 3} \left(3 + f(x - \frac{1}{2})\right) = 3 + 10 = 13.$$ 

4b) Let $h = x - 1$

$$\lim_{x \to 3} \frac{f(x + \frac{1}{2})}{x - 1} = \lim_{h \to 0} \frac{f(h + \frac{3}{2})}{h} = \lim_{h \to 0} \frac{f(h + \frac{3}{2}) - f\left(\frac{3}{2}\right)}{h} = f'\left(\frac{3}{2}\right) = 3.$$ 

5a) Differentiating both sides of the equation $x + \sin(xy) = y$ with respect to $x$, we find that

$$1 + \cos(xy) \left(y + x \frac{dy}{dx}\right) = \frac{dy}{dx}.$$ 

Solving for $y'(x) = \frac{dy}{dx}$, we find

$$\frac{dy}{dx} = \frac{1 + y \cos(xy)}{1 - x \cos(xy)} \quad \text{and in particular} \quad y'(0) = \frac{1 + 0 \cos(0)}{1 - 0 \cos(0)} = 1.$$ 

Therefore the equation for the tangent line is $y = 1(x - 0) + 0 = x$ and the equation of the normal line is $y = -1(x - 0) + 0 = -x$.

5b) Let $h$ be the length of the hypothenuse and $h^2 = x^2 + y^2$ with $\frac{dx}{dt} = -1 \text{ cm/min}$ and $\frac{dy}{dt} = 2 \text{ cm/min}$. Using implicit differentiation, we find that

$$2h \frac{dh}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt}.$$ 

At the instant when $x = 8$ and $y = 4$, we have $h = \sqrt{8^2 + 4^2} = 4\sqrt{5}$ and 

$$4\sqrt{5} \frac{dh}{dt} = 8(\cdot - 1) + 4.2 = 0 \quad \text{that is} \quad \frac{dh}{dt} = 0.$$