Induction and Sequences

Let \( n_0 \) be an integer. Suppose that we want to prove that a statement \( S(n) \) about integers \( n \geq n_0 \) is true for all \( n \geq n_0 \). One way of doing this is to use the Induction Method. If we can prove that

- \( S(n_0) \) is true, and
- For each \( k \geq n_0 \), if \( S(k) \) is true then \( S(k + 1) \) is true,

then it follows that \( S(n) \) is true for all \( n \geq n_0 \).

**Example 0:** Consider the sequence \( a_n = \frac{10^n}{n!} \) for \( n \geq 0 \). Then

\[
a_0 = 1, \quad a_1 = 10, \quad a_2 = 50, \quad a_3 = 500 \cdot \frac{3}{3}, \quad a_4 = 1250 \cdot \frac{3}{3}.
\]

In particular, \( a_0 < a_1 < a_2 < a_3 < a_4 \). Can we conclude that \( \{a_n\}_{n=0}^{\infty} \) is an increasing sequence? No, because, for instance,

\[
a_{10} = \frac{15262500}{567} > \frac{15625000}{6237} = a_{11}.
\]

In fact, we can prove that the sequence \( \{a_n\}_{n=10}^{\infty} \) is decreasing. If \( n \geq 10 \), then

\[
a_{n+1} = \frac{10^{n+1}}{(n+1)!} = \frac{10^n}{n!} \cdot \frac{10}{n+1} \leq \frac{10^n}{n!} \cdot \frac{10}{11} < \frac{10^n}{n!} = a_n.
\]

**Example 1:** Consider the sequence recursively defined by the conditions

\[
a_4 = 7 \quad \text{and} \quad a_n = \frac{a_{n-1} + 1}{4} \quad \text{for} \quad n \geq 2.
\]

We want to show that this sequence is convergent using the Monotonic Sequence Theorem. First we will show that the sequence is bounded below by 0.

**Claim:** \( a_n > 0 \) for all \( n \geq 1 \).

**Proof:** We will use induction.

- For \( n = 1 \), we have \( a_1 = 7 > 0 \).
- Suppose that \( a_k > 0 \) for some \( k \geq 1 \). Then

\[
a_k > 0 \implies a_k + 1 > 1 \implies a_{k+1} = \frac{a_k + 1}{4} > \frac{1}{4} > 0
\]

and we are done. \( \square \)

Next we will show that the sequence is decreasing.

**Claim:** \( a_{n+1} < a_n \) for all \( n \geq 1 \).

**Proof:** We will use induction again.

- For \( n = 1 \), we have \( a_2 = \frac{7 + 1}{4} = 2 < 7 = a_1 \).
- Suppose that \( a_{k+1} < a_k \) for some \( k \geq 1 \). We want to show that \( a_{k+2} < a_{k+1} \). This is true, because

\[
a_{k+1} < a_k \implies a_{k+1} + 1 < a_k + 1 \implies a_{k+2} = \frac{a_{k+1} + 1}{4} < \frac{a_k + 1}{4} = a_{k+1}.
\] \( \square \)
Finally, since the sequences is bounded from below and decreasing, it is convergent by the Monotonic Sequence Theorem.

**Example 2:** In this Example we will again show that the sequence defined by

\[ a_1 = 7 \quad \text{and} \quad a_n = \frac{a_{n-1} + 1}{4} \quad \text{for} \quad n \geq 2 \]

is convergent. But this time we will not use induction in the second part of the proof.

First we will show that the sequence is bounded below by 1/3.

**Claim:** \( a_n > 1/3 \) for all \( n \geq 1 \).

**Proof:** We will use induction.

- For \( n = 1 \), we have \( a_1 = 7 > 1/3 \).
- Suppose that \( a_k > 1/3 \) for some \( k \geq 1 \). Then:

\[ a_k > \frac{1}{3} \implies a_k + 1 > \frac{4}{3} \implies a_{k+1} = \frac{a_k + 1}{4} > \frac{1}{3} \]

Now we will prove that the sequence is decreasing. As we already know that \( a_n > 1/3 \) and therefore \( 1 - 3a_n < 0 \) for all \( n \geq 1 \), it immediately follows that

\[ a_{n+1} - a_n = \frac{a_n + 1}{4} - a_n = \frac{1}{4} (1 - 3a_n) < 0 \]

and hence \( a_{n+1} < a_n \) for all \( n \geq 1 \).

Once again, since the sequences is bounded from below and decreasing, it is convergent by the Monotonic Sequence Theorem.

**Example 3:** In this Example we will one more time show that the sequence defined by

\[ a_1 = 7 \quad \text{and} \quad a_n = \frac{a_{n-1} + 1}{4} \quad \text{for} \quad n \geq 2 \]

is convergent, but this time without using the Monotonic Sequence Theorem.

Note that

\[ a_{n+1} - \frac{1}{3} = \frac{a_n + 1}{4} - \frac{1}{3} = \frac{1}{4} \left( a_n - \frac{1}{3} \right) \]

for \( n \geq 1 \). Therefore, we obtain

\[ a_n - \frac{1}{3} = \frac{1}{4} \left( a_{n-1} - \frac{1}{3} \right) = \left( \frac{1}{4} \right) \left( a_{n-2} - \frac{1}{3} \right) = \cdots = \left( \frac{1}{4} \right)^{n-2} \left( a_2 - \frac{1}{3} \right) = \left( \frac{1}{4} \right)^{n-1} \left( a_1 - \frac{1}{3} \right) \]

and use this to conclude that

\[ \lim_{n \to \infty} \left( a_n - \frac{1}{3} \right) = \lim_{n \to \infty} \left( \left( \frac{1}{4} \right)^{n-1} \left( a_1 - \frac{1}{3} \right) \right) = \left( a_1 - \frac{1}{3} \right) \lim_{n \to \infty} \left( \frac{1}{4} \right)^{n-1} = 0 \]

as \( \lim_{n \to \infty} r^n = 0 \) for \( |r| < 1 \). Hence \( \lim_{n \to \infty} a_n = \frac{1}{3} \).

Two remarks about this proof: Firstly, note that this time we did not use the fact \( a_1 = 7 \) at all. That is, we proved that any sequence defined by this recursion relation converges to 1/3, no matter what the initial condition is. Secondly, there is an induction hidden somewhere in there. Find it and write it out explicitly as an exercise.