1)  
   a) If we put the D.E. in normal form, then $p(x) = x^{-1}$ and $q(x) = 1 - (1/9x^2)$. Thus $x = 0$ is a singular point. Since, $xp(x) \to 1$ and $x^2q(x) \to -1/9$ as $x \to 0$. It follows that $x = 0$ is a regular point. In determining a series solution of the D.E. it is better to leave the D.E. as it is given. If we substitute $y = \sum_{n=0}^{\infty} a_n x^{r+n}$, then we obtain

$$
\sum_{n=0}^{\infty} (n + r)(n + r - 1)a_n x^{r+n} + \sum_{n=0}^{\infty} (n + r)a_n x^{r+n} + \left(x^2 - \frac{1}{9}\right) \sum_{n=0}^{\infty} a_n x^{r+n} = 0
$$

After the shifting the summation index, we obtain the following

$$
\left[ r(r - 1) + r - \frac{1}{9} \right] a_0 x^r + \left[ (r + 1)r + (r + 1) - \frac{1}{9} \right] a_1 x^{r+1} + \sum_{n=2}^{\infty} \left\{ [(n + r)(n + r - 1) + (n + r) - \frac{1}{9}]a_n + a_{n-2} \right\} x^{n+r} = 0.
$$

The I.E. is

$$
r^2 - \frac{1}{9} = 0
$$

with roots $r_1 = 1/3$ and $r_2 = -1/3$. For either values of $r$ it is necessary to take $a_1 = 0$ in order that the coefficient of $x^{r+1}$ be zero. The recursion relation is

$$
\left[ (n + r)^2 - \frac{1}{9} \right] a_n = -a_{n-2}.
$$

for $r = 1/3$ we have

$$
a_n = \frac{-a_{n-2}}{(n + \frac{1}{3})^2 - (\frac{1}{3})^2} = -\frac{a_{n-2}}{(n + \frac{2}{3}) n}, \quad n = 2, 3, 4, \ldots
$$

Since $a_1 = 0$ from the recursion relation $a_3 = a_5 = a_7 = \ldots = 0$. For the even coefficients it is convenient to let $n = 2m, \quad m = 1, 2, 3, \ldots$ Then

$$
a_{2m} = -\frac{a_{m-2}}{2^m m + \frac{1}{3}}.
$$

The coefficient of $x^{2m}$ for $m = 1, 2, 3, \ldots$ is

$$
a_{2m} = \frac{(-1)^m a_0}{2^m m! \left(1 + \frac{1}{3}\right) \left(2 + \frac{1}{3}\right) \cdots \left(m + \frac{1}{3}\right)}.
$$

Thus one solution by setting $a_0 = 1$ is

$$
y_1(x) = x^{1/3} \left[ 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m! \left(1 + \frac{1}{3}\right) \left(2 + \frac{1}{3}\right) \cdots \left(m + \frac{1}{3}\right)} \left(\frac{x}{2}\right)^{2m} \right].
$$

---

\[I\text{made every effort to avoid the calculation errors and/or typos while I prepared the solution set. You are responsible to check all the solutions and correct the errors if there are any. If you find any errors and/or misprints, please notify me.}\]
Since \( r_2 = -1/3 \neq r_1 \) and \( r_1 - r_2 = 2/3 \) is not an integer. So we can calculate the second L.I. solution corresponding to \( r_2 = -1/3 \). The recursion relation is

\[
a_{n-2} = -n \left(n - \frac{2}{3}\right) a_n
\]

which yields the desired solution following the steps just outlined above. Note that \( a_1 = 0 \), as in the first solution, and thus all the odd coefficients are zero.

Thus one solution of the Bessel equation of order zero is

\[
y_2(x) = x^{-1/3} \left[ 1 - \sum_{n=1}^{\infty} \frac{(-1)^m}{m! \left(1 - \frac{1}{3}\right) \left(2 - \frac{1}{3}\right) \ldots \left(m - \frac{1}{3}\right)} \left(\frac{x}{2}\right)^{2m} \right].
\]

b) \( x = 0 \) is the regular singular point of the D.E. If we substitute \( y = \sum_{n=0}^{\infty} a_n x^{r+n} \), into D.E. and shifting the index yield the following indicial equation and the recursion relation

\[
r^2 = 0, \quad (r + n + 1)^2 a_{n+1} - a_n = 0, \quad n = 0, 1, 2, 3, \ldots
\]

So \( r = 0 \) is a double root. Thus we will obtain only one series of the form \( y = \sum_{n=0}^{\infty} a_n x^{r+n} \). For \( r = 0 \) the recursion recursion relation gives \( a_0 = a_1 \), \( a_2 = a_1/2^2 = a_0/2^2 \), \( a_3 = a_2/3^2 = a_0/2^2 \cdot 3^2 \), \( \ldots a_n = a_0/(n!)^2 \). Thus one solution, by setting \( a_0 = 1 \) is

\[
y = \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2}
\]

2) If we put the D.E. in normal form, then \( p(x) = x^{-1} \) and \( q(x) = 1 \). Thus \( x = 0 \) is a singular point. Since, \( xp(x) \to 1 \) and \( x^2 q(x) \to 0 \) as \( x \to 0 \). It follows that \( x = 0 \) is a regular point. In determining a series solution of the D.E. it is better to leave the D.E. as it is given. If we substitute \( y = \sum_{n=0}^{\infty} a_n x^{r+n} \), then we obtain

\[
\sum_{n=0}^{\infty} (n+r)(n+r-1)a_n x^{r+n} + \sum_{n=0}^{\infty} (n+r)a_n x^{r+n} + \sum_{n=2}^{\infty} a_{n-2} x^{r+n} = 0
\]

\[
[r(r-1) + r]a_0 x^r + [(1+r)r+1+r]a_1 x^{r+1} + \sum_{n=2}^{\infty} [(n+r)^2 a_n + a_{n-2}] x^{n+r} = 0
\]

The I.E. is \( r^2 = 0 \) so \( r = 0 \) is a double root. It is necessary to take \( a_1 = 0 \) in order that the coefficient of \( x^{r+1} \) be zero. The recursion relation is

\[
n^2 a_n = -a_{n-2}, \quad n = 2, 3, 4, \ldots
\]

Since \( a_1 = 0 \) it follows that \( a_3 = a_5 = a_7 = \ldots = 0 \). For the even coefficients we let \( n = 2m, \quad m = 1, 2, 3, \ldots \) Then

\[
a_{2m} = -\frac{a_{2m-2}}{2^2 m^2}
\]

So,

\[
a_2 = -\frac{a_0}{2^2}, \quad a_4 = \frac{a_0}{2^2 4^2}, \ldots, a_{2m} = (-1)^m \frac{a_0}{2^{2m} (m!)^2}.
\]

Thus one solution of the Bessel equation of order zero is

\[
J_0(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{x^{2m}}{2^{2m} (m!)^2}
\]
by setting $a_0 = 1$. Using the ratio test we can show that the the series converges for all $x$. Also note that $J_0(x) \to 1$ as $x \to 0$.

3)

a) If we put the D.E. in normal form, then $p(x) = x^{-1}$ and $q(x) = (x^2 - 1)/x^2$. Thus $x = 0$ is a singular point. Since, $xp(x) \to 1$ and $x^2q(x) \to -1$ as $x \to 0$. It follows that $x = 0$ is a regular point. In determining a series solution of the D.E. it is better to leave the D.E. as it is given. If we substitute $y = \sum_{n=0}^{\infty} a_n x^{r+n}$, shifting the summation indices appropriately and collecting coefficients of common powers of $x$ we obtain

$$[r(r - 1) + r - 1]a_0 x^r + [(1 + r)r + 1 + r - 1]a_1 x^{r+1} + \sum_{n=2}^{\infty} \{[(n + r)^2 - 1]a_n + a_{n-2}\} x^{r+n} = 0.$$ 

The I.E. $r^2 - 1 = 0$ so the roots are $r_1 = 1$ and $r_2 = -1$. For either value of $r$ it is necessary to take $a_1 = 0$ in order that the coefficient of $x^{r+1}$ be zero. The recursion relation

$$a_{n-2} = -a_n[(n + r)^2 - 1], \quad n = 2, 3, 4, ...$$

Since $a_1 = 0$ it follows that $a_3 = a_5 = a_7 = ... = 0$. For the even coefficients we let $n = 2m, \quad m = 1, 2, 3, ...$ Then

$$a_{2m} = -\frac{a_{2m-2}}{2^m(m + 1)}, \quad m = 1, 2, 3, ...$$

So,

$$a_2 = -\frac{a_0}{2^1}, \quad a_4 = -\frac{a_0}{2^22^1}, \quad ..., \quad a_{2m} = (-1)^m \frac{a_0}{2^{2m}m!(m + 1)!}.$$ 

Thus one solution of the Bessel equation of order one (by setting $a_0 = 1/2$)

$$J_1(x) = \frac{x}{2} \sum_{m=0}^{\infty} (-1)^m \frac{x^{2m}}{2^{2m}m!(m + 1)!}.$$ 

Using the ratio test we can show that the the series converges for all $x$. Also note that $J_1(x) \to 0$ as $x \to 0$.

b) For $r = -1$ the recursion relation is

$$[(n - 1)^2 - 1]a_n = -a_{n-2}, \quad n = 2, 3, ...$$

Substituting $n = 2$ into recursion relation we obtain

$$[(2 - 1)^2 - 1]a_2 = -a_0$$

Hence, it is not possible to determine $a_2$ and consequently not possible to find a series solution of the form $x^{-1} \sum_{n=0}^{\infty} b_n x^n$.

4)

a) If we write the equation in normal form then $p(x) = 2, \quad q(x) = 6e^x/x$. Thus $x = 0$ is a singular point of the D.E. Since $xp(x) = 2x$ and $x^2q(x) = 6xe^x$ are analytic at $x = 0$ so $x = 0$ is a regular singular point of the D.E. Next we have $xp(x) \to 0 = p_0$ and $x^2q(x) \to 0 = q_0$ as $x \to 0$. Thus the I.E. is $r(r - 1) + p_0r + q_0 = r^2 - r = 0$ which has roots $r_1 = 1$ and $r_2 = 0$.

b) The equation has the form of $P(x)y'' + Q(x)y' + R(x)y = 0$ with $P(x) = x(x-1), \quad Q(x) = 6x^2$
and \( R(x) = 3 \). Since \( P(x) \), \( Q(x) \) and \( R(x) \) are polynomials with no common factors and \( P(x) = 0 \) when \( x = 0, 1 \). Hence \( x = 0, 1 \) are the singular points of the D.E.

\( x = 0 \): Since, \( p(x) = 6x/(x-1) \), and \( q(x) = 3/(x-1) \), then \( xp(x) \to 0 = p_0 \) and \( x^2q(x) \to 0 = q_0 \) as \( x \to 0 \). Therefore, the I.E. is \( r(r-1) = 0 \) and the roots are \( r_1 = 0 \) and \( r_2 = 1 \).

\( x = 1 \): Since, \( p(x) = 6x/(x-1) \), and \( q(x) = 3/(x-1) \), then \( (x-1)p(x) = 6x \to 6 = p_0 \) and \( (x-1)^2q(x) = 3(x-1)/x \to 0 = q_0 \) as \( x \to 0 \). Therefore, the I.E. is \( r(r-1) + 6r = r(r+5) = 0 \) and the roots are \( r_1 = 0 \) and \( r_2 = -5 \).

c) For this D.E.

\[
p(x) = -\frac{1+x}{x^2(1-x)}, \quad q(x) = \frac{2}{x(1-x)}
\]

and thus \( x = 0 \) and \( x = -1 \) are the singular points. Since \( xp(x) \) is not analytic at \( x = 0 \). Hence, \( x = 0 \) is an irregular singular point. For \( x = -1 \),

\[
(x-1)p(x) = \frac{(1+x)}{x^2}, \quad (x-1)^2q(x) = \frac{2(1-x)}{x}
\]

are both analytic at \( x = -1 \). Therefore \( x = -1 \) is a regular singular point and that \( p_0 = 2 \) and \( q_0 = 0 \).

5) Write the D.E. in normal form, since we are looking for the solution about \( x = 1 \), multiply the D.E. with \((x-1)^2\) and get

\[
(x-1)^2y'' + (x-1) \left[ \frac{(x-1)}{2\ln x} \right] y' + \left[ \frac{(x-1)^2}{\ln x} \right] y = 0
\]

Since \( \ln 1 = 0 \), \( x = 1 \) is a singular point. To show it is a regular singular point of the D.E. we must show that \((x-1)/\ln x\) is analytic at \( x = 1 \) and \((x-1)^2/\ln x = (x-1)[(x-1)/\ln x]\) is also analytic at \( x = 1 \). We expand \( \ln x \) in a Taylor series about \( x = 1 \) we find that

\[
\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - ...
\]

Thus \( \frac{(x-1)}{\ln x} \) has the following Taylor series expansion about \( x = 1 \)

\[
\frac{(x-1)}{\ln x} = \left[ 1 - \frac{1}{2}(x-1) + \frac{1}{3}(x-1)^2 - ... \right]^{-1} = 1 + \frac{1}{2}(x-1) + ....
\]

and hence it is analytic. We can use the above result to obtain the indicial equation at \( x = 1 \). We have

\[
(x-1)^2y'' + (x-1) \left[ \frac{1}{2} + \frac{1}{4}(x-1) + ... \right] y' + \left[ (x-1) + \frac{1}{2}(x-1)^2 + ... \right] y = 0.
\]

Thus \( p_0 = 1/2 \) and \( q_0 = 0 \) and the I.E. is

\[
r(r-1) + \frac{r}{2} = 0
\]

Hence \( r_1 = 1/2 \) and \( r_2 = 0 \). In order to find the first three non-zero terms in a series solution corresponding to \( r_1 = 1/2 \), it is better to keep the D.E. in its original form and substitute the above power series for \( \ln x \):

\[
[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + ...]y'' + \frac{1}{2}y' + y = 0
\]
If we substitute

\[ y = a_0(x - 1)^{1/2} + a_1(x - 1)^{3/2} + a_2(x - 1)^{5/2} + ... \]

and collect the coefficients of like powers of \((x - 1)\) which are then set equal to zero. Then we obtain

\[ \frac{6a_1}{4} + \frac{9a_0}{8} = 0, \quad 5a_2 + \frac{5a_1}{8} - \frac{a_0}{12} = 0 \]

These equations yield

\[ a_1 = -\frac{3a_0}{4}, \quad a_2 = \frac{53a_0}{480} \]

with \(a_0 = 1\) we obtain the solution

\[ y_1(x) = (x - 1)^{1/2} - \frac{3}{4}(x - 1)^{3/2} + \frac{53}{480}(x - 1)^{5/2} + ... \]

Since the radius of convergence of the series for \(\ln x\) is 1, we would expect \(\rho = 1\).