Homework problems from the 2nd Edition, SECTION 1.3

12 (22) Since, \( f(x, y) = x \ln y \), and
\[
\frac{\partial f}{\partial y} = \frac{x}{y}
\]
are both continues in a neighborhood of \((1, 1)\), by the theorem of existence and uniqueness the solution exists and unique in some neighborhood of \((1, 1)\).

14 (24) Similar to previous problem, \( f(x, y) = y^{1/3} \) is continuous in a neighborhood of \((0, 0)\), but \( \partial f/\partial y = (1/3)y^{-2/3} \) is not, so the theorem guarantees existence but not uniqueness in some neighborhood of \( x = 0 \).

15 (25) \( f(x, y) = (x - y)^{1/2} \) is not continuous at \((2, 2)\) because it is not even defined if \( y > x \). Hence the theorem guarantees neither existence nor uniqueness in any neighborhood of the point \( x = 2 \).

17 (27) Since
\[
f(x, y) = \frac{x - 1}{y}
\]
and
\[
\frac{\partial f}{\partial y} = -\frac{x - 1}{y^2}
\]
are both continuous near the point \((0, 1)\), so the theorem guarantees both existence and uniqueness of a solution in some neighborhood of \( x = 0 \).

18 (28) Neither \( f(x, y) = (x - 1)/y \) nor \( \partial f/\partial y = -(x - 1)/y^2 \) is continuous near \((1, 0)\), so the existence-uniqueness theorem guarantees nothing.

20 (30) Both \( f(x, y) = x^2 - y^2 \) and \( \partial f/\partial y = -2y \) are continues near \((0, 1)\). So by the theorem of existence and uniqueness the solution exists and unique in some neighborhood of \( x = 0 \).

27) Just take the derivatives of the given \( y(x) \) for \( x \leq c \) and for \( x > c \) and substitute in to D.E., you will see that the D.E. is identically satisfied. So the given \( y(x) \) solve the D.E. If \( b < 0 \) then the initial value problem \( y' = 2\sqrt{y}, \ y(0) = b \) has no solution, because the square root of a negative number would be involved. If \( b > 0 \) we get a unique solution curve through \((0, b)\) defined for all \( x \) by following a parabola down (and leftward) to the \( x \)-axis and then following the \( x \)-axis to the left.

1 I made every effort to avoid the calculation errors and/or typos while I prepared the solution set. You are responsible to check all the solutions and correct the errors if there is any. If you find any errors and/or misprints, please notify me.

2 The number in the parenthesis denotes the problem number in the International Edition.
But starting at \((0,0)\) we can follow the positive \(x-\)axis to the point \((0,c)\) and then branching off on the parabola \(y = (x-c)^2\). This gives infinitely many different solutions if \(b = 0\).

28) I.V.P. \(xy' = y, \ y(a) = b\) has a unique solution off the y-axis where \(a \neq 0\); infinitely many solutions through the origin where \(a = b = 0\); no solution if \(a = 0\) but \(b \neq 0\) (so the point \((a,b)\) lies on the positive or negative \(y-\)axis).

29) If we sketch the graph of the solution, we see that we can start at the point \((a,b)\) and follow a branch of a cubic up or down to the \(x-\)axis, then follow the \(x-\)axis an arbitrary distance before branching off (down or up) on another cubic. This gives infinitely many solutions of the initial value problem \(y' = 3y^{2/3}, \ y(a) = b\) that are defined for all \(x\). However, if \(b \neq 0\) there is only a single cubic \(y = (x-c)^3\) passing through \((a,b)\), so the solution is unique near \(x = a\).

30) The function \(y = \cos(x-c)\) satisfies the given D.E. \(y' = -\sqrt{1-y^2}\) on the interval \(c < x < c + \pi\) where \(\sin(x-c) > 0\), so it follows that

\[-\sqrt{1-y^2} = -\sqrt{1-\cos^2(x-c)} = -\sqrt{\sin^2(x-c)} = -\sin(x-c) = y'.\]

If \(|b| > 1\) then the initial value problem

\[y' = -\sqrt{1-y^2}, \quad y(a) = b,\]

has no solution because the square root of a negative number would be involved. If \(|b| < 1\) then there is only one curve of the form \(y = \cos(x-c)\) through the point \((a,b)\); this gives a unique solution. But if \(b = \pm 1\) then we can combine a left ray of the line \(y = +1\), a cosine curve from the line \(y = +1\) to the line \(y = -1\), and then a right ray of the line \(y = -1\). If we sketch the graph of the solution, we see that this gives infinitely many solutions (defined for all \(x\)) through any point of the form \((a, \pm 1)\).

31) The function \(y = \sin(x-c)\) satisfies the D.E. on the interval \(c - (\pi/2) < x < c + (\pi/2)\) where \(\cos(x-c) > 0\), so it follows that

\[\sqrt{1-y^2} = \sqrt{1-\sin^2(x-c)} = \sqrt{\cos^2(x-c)} = -\sin(x-c) = y'.\]

If \(|b| > 1\) then the initial value problem

\[y' = \sqrt{1-y^2}, \quad y(a) = b\]

has no solution because the square root of a negative number would be involved. If \(|b| < 1\) then there is only one curve of the form \(y = \sin(x-c)\) through the point \((a,b)\); this gives a unique solution. But if \(|b| = \pm 1\) then we can combine a left ray of the line \(y = -1\), a sine curve from the line \(y = -1\) to the line \(y = +1\), and then a right ray of the line \(y = +1\). If we sketch the solution curve, we see that this gives infinitely many solutions (defined for all \(x\)) through any point of the form \((a, \pm 1)\).