Bäcklund transformations for discrete Painlevé equations : dP_{II}-dP_{V}

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Abstract. Transformation properties of discrete Painlevé equations are investigated by using an algorithmic method. This method yields explicit transformations which relate the solutions of discrete Painlevé equations, dP_{II}-dP_{V}, with different parameters. Rational solutions and elementary solutions of discrete Painlevé equations can also be obtained from these transformations.

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1. Introduction

Painlevé and his school classified the integrable second order equation of the form $y'' = f(x, y, y')$ where $f$ is rational in $y'$ and $y$ and analytic in $x$, whose solutions have no movable critical points, and discovered six transcendental equations that are called Painlevé equations, $P_I$ - $P_{VI}$ [1, 2, 3]. Their general solutions can not be expressed in terms of the known elementary functions and can be regarded as nonlinear analogues of the classical special functions. However, $P_{II}$ - $P_{VI}$ have rational solutions and solutions expressible in terms of the classical special functions for certain values of parameters. $P_{II}$ - $P_{VI}$ also possess Bäcklund transformations which relate solutions of the same equation with different values of parameters, or to solution of another equation of Painlevé type [4, 5, 6]. Although, Painlevé equations were first discovered from strictly mathematical considerations, they have appeared in physical applications. For example, $P_{III}$ arises in the Ising model [7], and $P_{IV}$ appears in quantum gravity [8].

Discrete analogues of the Painlevé equations are nonautonomous mappings that are integrable in the same sense as the continuous Painlevé equations [9, 10, 11], and recently have attracted much attention. The discrete Painlevé equations, $dP_I$-$dP_{VI}$, which have the form

$$x_{n+1} = \frac{f_1(x_n; n) + x_{n-1}f_2(x_n; n)}{f_3(x_n; n) + x_{n-1}f_4(x_n; n)}$$

(1)

where $f_j(x_n; n)$ are polynomials of degree at most four in $x_n$ [12]. In continuous limit, the discrete Painlevé equations yield a Painlevé equation, though some of the discrete Painlevé equations have limits more then one Painlevé equation. More over they share the common properties with their continuous counterparts, e.g. both posses Lax’s pairs, form coalescence cascades, have Bäcklund and Miura transformations, and have particular solutions in terms of special functions or rational solutions for special values of their parameters [13, 14]. Discrete Painlevé equations also appear in physics, for example, the computation of a certain partition function in a model of two-dimensional quantum gravity led $dP_I$ [15]. The only difference between continuous and discrete Painlevé equations is that the continuous Painlevé equations have unique canonical form up to a Möbius transformation, but there is more then one inequivalent discrete equation which has the Painlevé equation as its continuous limit.

In this article, we investigate the transformation properties of the discrete Painlevé equations by using an algorithmic method which is similar to the method developed by Fokas and Ablowitz [5] for investigating the transformation properties of the continuous equations of the Painlevé type. In [5], for given continuous Painlevé equations

$$v'' = P_2(v')^2 + P_1v' + P_3$$

(2)

where $P_1$, $P_2$, $P_3$ depend on $v$, independent variable $z$ and set of parameters $\alpha$, the transformation of type

$$u(z; \hat{\alpha}) = v'' + av^2 + bv + c$$

$$du^2 + ev + f$$

(3)

where $a, b, ..., d$ depend on $z$ only and $u(z; \hat{\alpha})$ solves some second order equation of the Painlevé type with set of parameters $\hat{\alpha}$, was considered. If we solve (3) for $v'$, we obtain

$$v' = (du - a)v^2 + (eu - b)v + fu - c.$$  

(4)
That is, solution $v$ of the given equation of Painlevé type also satisfies a Riccati equation with the coefficients depending linearly on the solution $u$ of related Painlevé type equation. By following the similar argument, for a given discrete Painlevé equation (1) with parameter set $\alpha$, we consider discrete Riccati equation, that is

$$x_{n+1} = \frac{A_n x_n + B_n}{C_n x_n + D_n}$$  \hspace{1cm} (5)$$

where $A_n = A_1, n y_n + A_0, n$, $B_n = B_1, n y_n + B_0, n$, $C_n = C_1, n y_n + C_0, n$, and $D_n = D_1, n y_n + D_0, n$ such that $y_n$ solves discrete equation of Painlevé type with parameter set $\hat{\alpha}$. The aim is to determine $A_j, n, ..., D_j, n$, $j = 0, 1$ requiring that (5) defines a one-to-one invertible map between the solutions $x_n$ of a given discrete Painlevé equation, and solutions $y_n$ of some second order discrete equation of Painlevé type. This method yields explicit transformations between a given discrete Painlevé equation and the same discrete Painlevé equation but with different values of its parameters, and between two different discrete equations of Painlevé type. As an application of the method, we obtain particular solutions of discrete Painlevé equations in terms of discrete analogue of the classical functions.

The method can be summarized as follows: From equation (5), one writes

$$x_{n-1} = -\frac{D_n x_n - B_{n-1}}{C_n x_n - A_{n-1}}.$$  \hspace{1cm} (6)$$

Substituting $x_{n+1}$ and $x_{n-1}$ given in (5) and (6) respectively into given discrete Painlevé equation (1) gives an equation which is polynomial for $x_n$ with the coefficients depending on $A_j, n, ..., D_j, n$, $j = 0, 1$, $y_n$ and $y_{n-1}$. Now, we choose $A_j, n, ..., D_j, n$ such that the polynomial for $x_n$ reduces to a polynomial of degree one or of degree two. That is,

$$E(y_n, y_{n-1}; n)x_n + F(y_n, y_{n-1}; n) = 0,$$  \hspace{1cm} (7)$$

or

$$E(y_n, y_{n-1}; n)x_n^2 + F(y_n, y_{n-1}; n)x_n + G(y_n, y_{n-1}; n) = 0.$$  \hspace{1cm} (8)$$

If one solves (7) or (8) for $x_n$, and substitutes into (5), (6) yields a discrete equation of Painlevé type for $y_n$. It turns out that, similar to the case of continuous Painlevé equations, dP\textsubscript{II}-dP\textsubscript{V} admit transformations of both types (7) and (8). However, dP\textsubscript{V} does not admit a transformation of type (7). In this article we will consider the transformation of type (7), the type (8) will be considered later and published elsewhere.

2. Discrete Painlevé II Equation

In this section, we consider dP\textsubscript{II} equation [16]

$$x_{n+1} + x_{n-1} = \frac{z_n x_n + a}{1 - x_n^2},$$  \hspace{1cm} (9)$$

where $z_n = \alpha n + \beta$, and $\alpha$, $\beta$, $a$ are constants. Substituting $x_{n+1}$ and $x_{n-1}$ given in (5) and (6) respectively into (9) gives the following polynomial for $x_n$, $$(z_n x_n + a)(C_n x_n + D_n)(C_{n-1} x_n - A_{n-1}) = (1 - x_n^2)(A_n x_n + B_n)(C_{n-1} x_n - A_{n-1}) - (C_n x_n + D_n)(D_{n-1} x_n - B_{n-1})$$  \hspace{1cm} (10)$$
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Our goal now is to choose \(A_n, \ldots D_n\) in such a way that (10) becomes a linear equation for \(x_n\). (10) reduces to a linear equation for \(x_n\),

\[
(B_n + B_{n-1} + z_n - 2)x_n = B_n - B_{n-1} - a
\]

with the choice of \(A_n = C_n = D_n = 1\). Without loss of generality, we may choose \(B_n = y_n - \frac{1}{2}z_n - \frac{1}{4}\alpha + 1\). Hence, equations (5) and (11) respectively give

\[
y_n = (x_n + 1)(x_{n+1} - 1) + \frac{1}{2}z_n + \frac{1}{4}\alpha
\]  

and

\[
x_n = \frac{y_n - y_{n-1} + \nu}{y_n + y_{n-1}},
\]

where \(\nu = -\frac{1}{2}\alpha - a\). Eliminating \(x_n\) between (12) and (13) leads to a discrete form of \(P_{XXXIV}[17, 18]\)

\[
(y_n + y_{n-1})(y_n + y_{n+1}) = \frac{\nu^2 - 4y_n^2}{y_n - \frac{1}{2}z_n - \frac{1}{4}\alpha}.
\]

2.1. Bäcklund Transformation for \(dP_{II}\)

Since, (14) is quadratic in \(\nu\), thus \(y_n(\nu) = y_n(-\nu)\). But then, from equation (13)

\[
\bar{x}_n(\nu) = \frac{y_n(-\nu) - y_{n-1}(-\nu) - \nu}{y_n(-\nu) - y_{n-1}(-\nu)}
\]

\[
= \frac{y_n(\nu) - y_{n-1}(\nu) - \nu}{y_n(\nu) + y_{n-1}(\nu)}
\]

\[
= x_n(\nu) - \frac{2\nu}{y_n(\nu) + y_{n-1}(\nu)}
\]

Hence, expressing \(y_n\) in terms of \(x_n\) and using \(\nu = -\frac{1}{2}\alpha - a\) give the following Bäcklund transformation for \(dP_{II}\)

\[
\bar{x}_n = x_n - \frac{2\nu(x_n + 1)}{2(x_{n+1} + 1) - z_nx_n - a}; \quad \bar{a} = -a - \alpha.
\]

This transformation was also given in [17, 18, 19].

2.2. Special Solution

The transformation (13) breaks down if

\[
y_n + y_{n-1} = 0
\]  

and

\[
y_n - y_{n-1} + \nu = 0.
\]

By solving (17) and (18), we find that \(y_n = \nu = 0\). Substituting \(y_n = \nu = 0\) into (12) yields the following discrete Riccati equation:

\[
x_{n+1} = \frac{2x_n - z_n + a + 2}{2(x_n + 1)}.
\]

Therefore, particular solution of \(dP_{II}\) is characterized by (19), iff \(a = -\frac{1}{2}\alpha\). (19) can be linearized by a Cole-Hopf transformation \(x_n = \frac{w_n}{w_{n-1}} - 1\), and we thus obtain the discrete analogue of the Airy equation [17, 18]:

\[
2w_{n+1} + (z_n - a)w_{n-1} - 4w_n = 0.
\]
3. Discrete Painlevé III Equation

In this section, we consider dP$_{III}$ equation [16]

\[
x_{n+1}x_{n-1} = \frac{ab(x_n - c\lambda^n)(x_n - d\lambda^n)}{(x_n - a)(x_n - b)}, \tag{21}
\]

where $a$, $b$, $c$, $d$ are constants. Using the method introduced in the introduction, we find

\[
\frac{(A_n x_n + B_n)(D_{n-1} x_n - B_{n-1})}{(C_n x_n + D_n)(C_{n-1} x_n - A_{n-1})} = \frac{ab(x_n - c\lambda^n)(x_n - d\lambda^n)}{(x_n - a)(x_n - b)}. \tag{22}
\]

With the choice of $D_n = -aC_n$ and $A_n = bC_n$, equation (22) can be reduced to the following linear equation for $x_n$:

\[\bbox{\text{III}} \quad [bC_nB_{n-1} + aC_{n-1}B_n + ab(c + d)\lambda^{2n}C_nC_{n-1}] x_n = abcd\lambda^{2n}C_nC_{n-1} - B_nB_{n-1}. \tag{23}\]

Without loss of generality we may let $B_n = \mu\lambda^{n+\frac{1}{2}}y_n$, $C_n = 1$, where $\mu^2 = abcd$. Then equations (5) and (23) become

\[
y_n = x_{n+1}(x_n - a) - bx_n \quad \mu\lambda^{n+\frac{1}{2}} \tag{24}
\]

and

\[
x_n = \frac{\mu^2\lambda^{n+\frac{1}{2}}(1 - y_n y_{n-1})}{a\mu\lambda y_n + b\mu y_{n-1} + ab\lambda\frac{1}{2}(c + d)} \tag{25}
\]

respectively. Eliminating $x_n$ between the equations (24) and (25) leads to the following discrete equation for $y_n$

\[
(y_n y_{n+1} - 1)(y_n y_{n-1} - 1) = -\frac{\lambda^n(y_n - \alpha)(y_n - \frac{1}{2})(y_n - \beta)(y_n - \frac{1}{2})}{(\gamma y_n - \lambda^n)}, \tag{26}
\]

where

\[
\alpha = -\frac{\mu}{ac\lambda^\frac{1}{2}} , \quad \beta = -\frac{\mu}{ad\lambda^\frac{1}{2}} , \quad \gamma = -\frac{\mu\lambda\frac{1}{2}}{ab} , \quad \lambda = \lambda^{-1}. \tag{27}
\]

Equation (26) is the special case, $\delta = 0$, of dP$_V$

\[
(y_n y_{n+1} - 1)(y_n y_{n-1} - 1) = \frac{\lambda^{2n}(y_n - \alpha)(y_n - \frac{1}{2})(y_n - \beta)(y_n - \frac{1}{2})}{(\gamma y_n - \lambda^n)(\delta y_n - \lambda^n)} \tag{28}
\]

Thus there exists the one to one correspondence given by (24) and (25) between the solutions of dP$_{III}$ and (26).

3.1. Bäcklund Transformation for dP$_{III}$

Bäcklund transformation can be obtained by finding two sets of $\{\alpha, \beta, \gamma\}$ such that (26) is invariant. It should be noted that equation (26) is invariant under the change of parameters

\[
\alpha = \frac{1}{\tilde{\alpha}} , \quad \beta = \frac{1}{\tilde{\beta}} , \quad \gamma = \gamma. \tag{29}
\]

By using (29) and following the same procedure given in section 2.1, we obtain the following Bäcklund transformation for dP$_{III}$

\[
x_n = \frac{d\lambda x_n[a x_{n+1}(x_n - a) + bx_{n-1}(x_n - b)] - 2abx_n + ab(c + d)\lambda^n}{c[bx_{n+1}(x_n - a) + a\lambda^2 x_{n-1}(x_n - b) - (a^2\lambda^2 + b^2)x_n + ab(c + d)\lambda^{n+1}]}, \tag{30}
\]

\[
\tilde{\alpha} = \frac{bd}{\lambda c} , \quad \tilde{\beta} = \frac{a\lambda d}{c} , \quad \tilde{\gamma} = \frac{d d}{c}.
\]
3.2. Special Solution

The transformation (25) breaks down if
\[ y_n y_{n-1} - 1 = 0 \]  
(31)
and
\[ a \mu \lambda y_n + b \mu y_{n-1} + a b \lambda^2 (c + d) = 0. \]  
(32)
By solving equations (31) and (32), we find that \( y_n = -\frac{bc}{\mu \sqrt{\lambda}}, \) and \( \lambda \mu^2 = b^2 c^2. \) Then, (24) leads to the following discrete Riccati equation [21],
\[ x_{n+1} = \frac{b(x_n - c \lambda^2)}{x_n - a}. \]  
(33)
Therefore, particular solution of \( \text{dP}_{III} \) is characterized by (33), iff \( \lambda \alpha d = bc. \) (19) can be linearized by a Cole-Hopf transformation \( x_n = a + \frac{w_n}{w_{n-1}}, \) and we thus obtain the discrete analogue of the Bessel equation [20]:
\[ w_{n+1} - b(a - c \lambda^2)w_{n-1} + (a - b)w_n = 0. \]  
(34)

4. Discrete Painlevé IV Equation

In this section, we consider \( \text{dP}_{IV} \) equation [16]
\[ (x_{n+1} + x_n)(x_{n+1} - x_n) = \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n - z_n)^2 - c^2}, \]  
(35)
where \( z_n = \alpha n + \beta, \) and \( a, b, \alpha, \beta \) are constants. Equation (35) gives the following equation after substituting \( x_{n+1} \) and \( x_{n-1} \) respectively given in (5) and (6),
\[ \frac{[C_n x_n^2 + (D_n + A_n)x_n + B_n][C_{n-1} x_n^2 - (D_{n-1} + A_{n-1})x_n + B_{n-1}]}{(C_n x_n + D_n)(C_{n-1} x_n - A_{n-1})} - \frac{(x_n^2 - a^2)(x_n^2 - b^2)}{(x_n - z_n)^2 - c^2}. \]  
(36)
(36) can be reduced to a linear equation for \( x_n \)
\[ x_n = \frac{-(y_n - y_{n-1} + \mu + \alpha)(y_n - z_n - \mu - \alpha) + z_n^2 - c^2}{y_n + y_{n-1}}, \]  
(37)
with the choice of \( A_n = -\frac{1}{2}(y_n - z_n + \mu), B_n = ab, C_n = 1, \) and \( D_n = y_n - z_n - \mu - \alpha, \) where \( \mu = -\frac{1}{2}(a + b + \alpha). \) With these choices, the equation (5) yields
\[ x_{n+1} = \frac{-(y_n - z_n + \mu)x_n + ab}{x_n + y_n - z_n - \mu - \alpha}. \]  
(38)
By eliminating \( x_n \) between (37) and (38), we obtain \( \text{dP}_{IV} \)
\[ (y_{n+1} + y_n)(y_{n-1} + y_n) = \frac{(y_n^2 - \tilde{a}^2)(y_n^2 - \tilde{b}^2)}{(y_n - z_n)^2 - c^2}, \]  
(39)
where
\[ \tilde{z}_n = z_n + \frac{1}{2} \alpha, \quad \tilde{a}^2 = \left(c - \frac{1}{2} (a + b - \alpha) \right)^2, \quad \tilde{b}^2 = \left[c + \frac{1}{2} (a + b - \alpha) \right]^2, \quad \tilde{c}^2 = \frac{1}{4} (a - b)^2. \]  
(40)
If we replace \( y_n \) with \( \tilde{x}_n \) in (38), we thus obtain the following Bäcklund transformation for \( \text{dP}_{IV} \)
\[ \tilde{x}_n = \frac{[x_{n+1}(x_n - z_n - \mu - \alpha) - x_n(z_n - \mu - \alpha - ab)]}{(x_{n+1} + x_n)}, \]  
(41)
such that \( \tilde{x}_n \) solves \( \text{dP}_{IV} \) with the parameters \( \tilde{a}, \tilde{b}, \tilde{c} \) given by (40).
4.1. Special Solution

The transformation (37) breaks down if
\[
y_n + y_{n-1} = 0
\]
and
\[
-(y_{n-1} - z + \mu + \alpha)(y_n - z_{n-1} - \mu - \alpha) + z^2 - c^2 = 0.
\]
y_n = \mu + \alpha + c = 0 solve the equations (42) and (43). Equation (38) yields the following discrete Riccati equation [21],
\[
x_{n+1} = \frac{(a + b - c + z_n)x_n + ab}{x_n + c - z_n},
\]
after substituting \(y_n = \mu + \alpha + c = 0\). Therefore, particular solution of dP_{V} satisfies (44), if \(a + b - 2c = \alpha\). Cole-Hopf transformation \(x_n = z_n - c + (w_n/w_{n-1})\) transforms the Riccati equation (44) into the following linear equation for \(w_n\):
\[
w_{n+1} - (z_n - c + a)(z_n - c + b)w_{n-1} - 2cw_n = 0.
\]

5. Discrete Painlevé V Equation

We consider the dP_v equation [16]
\[
(x_n, x_{n+1} - 1)(x_n, x_{n-1} - 1) = \frac{\lambda^{2n}(x_n - a)(x_n - b)(x_n - \frac{1}{a})(x_n - \frac{1}{b})}{(x_n - \lambda^n)(dx_n - \lambda^n)},
\]
where \(a, b, c, d\) are constants. Applying the method introduced in the introduction, we find
\[
\frac{[A_n x_n^2 + (B_n - C_n)x_n - D_n][(-D_{n-1}x_n^2 + (B_{n-1} - c_{n-1})x_n + A_{n-1})]}{(C_n x_n + D_n)(C_n x_n - A_{n-1})} = \frac{\lambda^{2n}(x_n - a)(x_n - b)(x_n - \frac{1}{a})(x_n - \frac{1}{b})}{(x_n - \lambda^n)(dx_n - \lambda^n)}.
\]
Equation (47) can be reduced to a linear equation for \(x_n\) with the choice of
\(A_n = \mu \lambda^n, B_n = y_n - (a + b)\mu \lambda^n, C_n = y_n\) and \(D_n = -ab A_n\), where \(\mu^2 = \frac{\lambda}{abcd}\). Then the equations (47) and (5) yield
\[
x_n = \frac{\mu \lambda^{n-1}(ab \lambda y_{n-1} + y_n) - (\frac{1}{a} + \frac{1}{b})\lambda^n}{y_n y_{n-1} - 1}
\]
and
\[
x_{n+1} = \frac{\mu \lambda^n (x_n - a - b) + y_n}{y_n x_n - ab \mu \lambda^n}.
\]
respectively. We obtain the following dP_v for \(y_n\) by eliminating \(x_n\) between the equations (48) and (49):
\[
(y_n, y_{n+1} - 1)(y_n, y_{n-1} - 1) = \frac{\lambda^{2n}(y_n - \alpha)(y_n - \frac{1}{\alpha})(y_n - \beta)(y_n - \frac{1}{\beta})}{(\gamma y_n - \lambda^n)(\delta y_n - \lambda^n)},
\]
where \(\gamma = \frac{1}{\alpha}, \delta = \frac{1}{\beta}, \alpha = \frac{\mu c}{\lambda}, \) and \(\beta = \frac{\mu d}{\lambda}\). Therefore, we have the following Bäcklund transformation for dP_v
\[
\begin{align*}
\bar{x}_n &= \frac{\mu \lambda^n [ab x_{n+1} + x_n - a - b]}{x_n x_{n+1} - 1}; \\
\bar{a} &= \frac{\mu c}{\lambda}, \quad \bar{b} = \frac{\mu d}{\lambda}, \quad \bar{c} = \frac{1}{a \mu}, \quad \bar{d} = \frac{1}{b \mu}.
\end{align*}
\]
5.1. Special Solution

The transformation (48) breaks down if
\[ y_n y_{n-1} - 1 = 0 \]  
and
\[ \mu \lambda^{n-1} (ab \lambda y_{n-1} + y_n) - \left( \frac{1}{c} + \frac{1}{d} \right) \lambda^n = 0. \]

If we substitute the solutions \( y_n = \frac{\lambda}{\mu c} \), and \( \lambda^2 = \mu^2 c^2 \) of (52) and (53) into (49), we obtain the following discrete Riccati equation [21],
\[ x_{n+1} = \frac{\lambda^n x_n - a - b}{ab(dx_n - \lambda^n)}. \]  

Therefore, particular solution of dP\(_V\) is characterized by (54), iff \( c = \lambda abd \). (19) can be linearized by a Cole-Hopf transformation \( x_n = \frac{\lambda^n}{d} + \frac{w_n}{w_{n-1}} \), and we thus obtain the following linear equation for \( w_n \)
\[ abd^2 w_{n+1} = (\lambda^n - ad)(\lambda^n - bd)w_{n-1} + \lambda^n(c - d)w_n = 0. \]

6. Conclusion

In this paper, we have presented the Bäcklund transformations for dP\(_{II}\)-dP\(_V\), by investigating the discrete Riccati type of transformation (5), such that the coefficients depend on the solution of a discrete equation of Painlevé type. The Miura transformation for dP\(_{II}\) and dP\(_{XXXIV}\), and the Bäcklund transformation for dP\(_{II}\) were known before. But the transformations for dP\(_{III}\)-dP\(_V\) were not discussed in the literature before. Moreover, we presented the particular solutions of dP\(_{II}\)-dP\(_V\) which exists whenever the parameters satisfy certain conditions, i.e. the linearisability conditions.

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References

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