Q1. Assume S is a nonempty subset of \( \mathbb{R} \) and \( c \) be a real constant. We define 
\[ c + S = \{ c + x : x \in S \} \]. Show that if \( S \) is bounded above then \( c + S \) is also bounded 
above and \( \sup(c + S) = c + \sup S \).

**Proof.** Since \( S \) is bounded above and \( S \neq \emptyset \), 
\[ m = \sup S \text{ exists in } \mathbb{R} \]. 
Let \( y \in c + S \), then \( \exists x \in S \text{ s.t. } y = c + x \). 
\[ x \in S \Rightarrow x \leq m, \text{ so } c + x \leq c + m, \text{ i.e. } y \leq c + m. \]

i) So \( c + S \) is bounded above and \( c + m \) is an upper bound for \( S \).

ii) Next given \( \epsilon > 0 \), \( \exists x \in S \text{ s.t. } m - \epsilon < x \). So 
\[ c + m - \epsilon < c + x \]. 
\[ y \in \text{an element of } c + S \]
So for every \( \epsilon > 0 \), there is an element \( y \in c + S \text{ s.t. } \]
\[ c + m - \epsilon < y. \]

i) and ii) \( \Rightarrow \) \( \sup(c + S) = c + m = c + \sup S \).

Instead of ii), we can also show the following.

ii') Let \( k \) be any upper bound for \( c + S \).
Now given any \( x \in S \), \( c + x \in c + S \Rightarrow c + x \leq k \)
\[ \Rightarrow x \leq k - c. \text{ That is, } k - c \text{ is an upper bound for } S. \]
Since \( m \) is the least upper bound for \( S \), we have \( m \leq k - c \Rightarrow c + m \leq k. \)

Then ii) and ii') also \( \Rightarrow \) \( \sup(c + S) = c + m = c + \sup S \).
Q2. Let \( X \) be the set of all finite subsets of \( \mathbb{N} \). Show that \( X \) is countable.

\[ \text{Proof.} \text{ Given } n = 0, 1, 2, \ldots, \text{ let } X_n \text{ be the set of all subsets of } \mathbb{N} \text{ with exactly } n \text{ elements. So } \\
X_0 = \{\emptyset\}, \quad X_1 = \{\{1\}, \{2\}, \{3\}, \ldots\}, \\
X_2 = \{\{1, 2\}, \{1, 3\}, \ldots, \{2, 3\}, \{2, 4\}, \ldots\}, \ldots \\
\]

We show that every \( X_n \) is countable, then \( X = \bigcup_{n=0}^{\infty} X_n \) being a countable union of countable sets is countable.

Clearly \( X_0 \) is finite, \( X_1 \) is countable.

For \( X_2 \)

\[
\begin{array}{c|cccc}
1 & 2 & 3 & 4 & \cdots \\
\hline
1 & \{1\} & \{1, 2\} & \{1, 3\} & \{1, 4\} & \cdots \\
2 & \{2\} & \{2, 3\} & \{2, 4\} & \cdots \\
3 & \{3\} & \{3, 2\} & \{3, 4\} & \cdots \\
4 & \{4\} & \{4, 2\} & \{4, 3\} & \{4, 4\} & \cdots \\
\end{array}
\]

Write the above sets as a sequence by using Cantor's method.

\[
\{1\}, \{1, 2\}, \{2, 1\}, \{1, 3\}, \{2, 1\}, \{3\}, \{1, 4\}, \{2, 3\}, \ldots
\]

in \( X_1 \) repeated

Drop the ones which have been already counted in \( X_1 \) and also the ones which have been considered before (e.g. \( \{2, 1\} \) is dropped, because \( \{1, 2\} \) is considered before). This way elements of \( X_2 \) can be written as a sequence, say \( X_2 = \{S_1, S_2, S_3, \ldots\} \) where \( S_1 = \{1, 2\}, S_2 = \{1, 3\}, S_3 = \{1, 4\}, S_4 = \{2, 3\}, \ldots \).
For $X_3$, we have the following table:

<table>
<thead>
<tr>
<th></th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>${1} U S_1$</td>
<td>${1} U S_2$</td>
<td>${1} U S_3$</td>
<td>${1} U S_4$</td>
</tr>
<tr>
<td>2</td>
<td>${2} U S_1$</td>
<td>${2} U S_2$</td>
<td>${2} U S_3$</td>
<td>${2} U S_4$</td>
</tr>
<tr>
<td>3</td>
<td>${3} U S_1$</td>
<td>${3} U S_2$</td>
<td>${3} U S_3$</td>
<td>${3} U S_4$</td>
</tr>
<tr>
<td>4</td>
<td>${4} U S_1$</td>
<td>${4} U S_2$</td>
<td>${4} U S_3$</td>
<td>${4} U S_4$</td>
</tr>
</tbody>
</table>

Write the above sets as a sequence by using Cantor's method:

$\{1 \times 1\}, \{1 \times 2\}, \{2 \times 1\}, \{1 \times 3\}, \{2 \times 2\}, \{3 \times 1\}, \ldots$

in $X_2$ in $X_2$ in $X_2$ in $X_2$ in $X_2$, repeated

Drop the ones which have been already counted in $X_2$ and also the ones which have been considered before.
This way elements of $X_2$ can be written as a sequence. We continue this way (in fact there is an induction step here).

We see that every $X_n$ is countable (except $X_0$ which is finite), then $\bigcup_{n=1}^{\infty} X_n$ is also countable, and adding the finite set $X_0$ to this union we get $X = \bigcup_{n=0}^{\infty} X_n$, which is also countable.
Q3. For the following sets $E$, find $\text{int}(E), E'$ and $\overline{E}$:

a) In $\mathbb{R}$, $E = (-\infty, 1) \cup [3, 4] \cup \{4\}$

$\text{int}(E) = (-\infty, 1) \cup (1, 3)$

$E' = (-\infty, 3] \quad (1 \text{ and } 3 \text{ are in } E')$

$\overline{E} = (-\infty, 3] \cup \{4\}$

b) In $\mathbb{R}^2$, $E = \{(x, y) : x^2 + y^2 < 1\} \cup \{(x, y) : (x - 1)^2 + y^2 < 1\}$

$\text{int}(E) = \{(x, y) : x^2 + y^2 < 1\} \cap \{(x, y) : (x - 1)^2 + y^2 < 1\}$

$E' = E = \{(x, y) : x^2 + y^2 \leq 1\} \cap \{(x, y) : (x - 1)^2 + y^2 \leq 1\}$
4. Let $E$ and $F$ be two nonempty subsets of a metric space. Show that

$$\text{int}(E \cap F) = \text{int}(E) \cap \text{int}(F).$$

**Proof.** Step 1. \(\text{int}(E \cap F) \subseteq \text{int}(E) \cap \text{int}(F)\)

Let \(p \in \text{int}(E \cap F)\). Then there is \(r > 0\) s.t.

\(B_r(p) \subseteq E \cap F\). Then

\(B_r(p) \subseteq E\) and \(E \subseteq F \Rightarrow B_r(p) \subseteq E\). So \(p \in \text{int}(E)\).

Thus \(p \in \text{int}(E) \cap \text{int}(F)\).

Step 2. \(\text{int}(E) \cap \text{int}(F) \subseteq \text{int}(E \cap F)\)

Let \(p \in \text{int}(E) \cap \text{int}(F)\). Then \(p \in \text{int}(E)\) and \(p \in \text{int}(F)\).

Let \(r_1 = \text{dist}(E, F) \geq 0\). Then

\(B_r(p) \subseteq B_{r_1}(p)\) and \(B_r(p) \subseteq E \Rightarrow B_r(p) \subseteq E\)

\(\Rightarrow B_r(p) \subseteq B_{r_2}(p)\) and \(B_r(p) \subseteq F \Rightarrow B_r(p) \subseteq F\)

Thus \(B_r(p) \subseteq E \cap F\). So \(p \in \text{int}(E \cap F)\).
5. Let $(X, d)$ be a metric space. Let $E$ and $G$ be two subsets of $X$ where $G$ is open. Prove the following:

$$E \cap G = \emptyset \iff \overline{E} \cap G = \emptyset$$

**Proof:**

($\Rightarrow$): Assume $\overline{E} \cap G = \emptyset$. Since $E \subseteq \overline{E}$, we have $E \cap G \subseteq \overline{E} \cap G$. Hence, $E \cap G = \emptyset$.

($\Leftarrow$): Assume $E \cap G = \emptyset$. Then $E \subseteq \overline{E} \cap G$, which is open, so $\overline{E} \cap G$ is closed. Since $\overline{E}$ is the smallest closed set which contains $E$ and $G^c$ is a closed set which contains $E$, we have $\overline{E} \cap G^c$. Therefore, $\overline{E} \cap G = \emptyset$. 
