Question 1. (20 points) Let $X = \mathbb{R}^2$ with the Euclidean metric $d$, and let 

$$E = \{(x, y) : 0 < x \leq 2, \ 0 < y \leq 1\}.$$

By using the definition (i.e., without using the Heine-Borel Theorem), prove that $E$ is not compact.

Question 2. (20 points) Let $X = \mathbb{R}$ with the absolute value metric $d$. Let 

$$E = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\} \cup \{0\}.$$

By using the definition (i.e., without using the Heine-Borel Theorem), prove that $E$ is compact.

Question 3. (20 points) Let $X$ be any infinite set, and $d$ be the discrete metric on $X$. Prove that a subset $K$ of $X$ is compact $\iff$ $K$ is a finite set.

Question 4. (20 points) Remember given two nonempty subsets $A$ and $B$ of a metric space $(X, d)$, we define the distance $\text{dist}(A, B)$ between $A$ and $B$, by

$$\text{dist}(A, B) = \inf\{d(p, q) : p \in A, q \in B\}.$$

(a) Prove that in an arbitrary metric space $X$, there are sequences $\{p_n\}$ in $A$, and $\{q_n\}$ in $B$ such that $\lim_{n \to \infty} d(p_n, q_n) = \text{dist}(A, B)$.

(b) Assume now that $X = \mathbb{R}^2$ with the Euclidean metric $d$. Prove that if $A$ and $B$ are two nonempty subsets of $X$, such that $A$ is closed, and $B$ is compact, then there are points $p_0 \in A$ and $q_0 \in B$ such that $\text{dist}(A, B) = d(p_0, q_0)$.

Remark. If $A$ and $B$ are both closed, but neither is compact then the claim in (b) is not correct.