MATH 215 Homework 12

Turn in by May 16th, 2014 until 10:30 am.

Question 1. (20 × 2 = 40 points) Consider $\mathbb{R}$ with the absolute value metric.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \leq M$ for some constant $M < 1$. Prove that $f$ is a contraction. 
Hint. Use the Mean Value Theorem that you have seen in Calculus.

(b) Let $c$ be a constant such that $0 < c < 1$, and define $f : \mathbb{R} \to \mathbb{R}$ by 
$$f(x) = c \left( \frac{\cos^3 x}{3} - \cos x \right).$$
Prove that there is a unique point $x \in \mathbb{R}$ such that $f(x) = x$.

Question 2. (20 points) Let $(X,d)$ be a compact metric space, and $f : X \to X$ be any function such that $d(f(p), f(q)) < d(p,q)$ for all $p,q \in X$ such that $p \neq q$. The purpose of this problem is to prove that $f$ has a unique fixed point. You’re suggested to proceed as follows:

- Define $g : X \to \mathbb{R}$ by $g(p) = d(p, f(p))$. Prove that $g$ is continuous on $X$.
- Let $a \in X$ be a point such that $g(a)$ is the minimum of $g(p)$ for $p \in X$. Why does such a point $a \in X$ exist?
- Prove that $a$ is a fixed point of $f$.
- Prove that there is no other fixed point of $f$.

Question 3. (10 + 30 = 40 points)

(a) Let $CN[a,b]$ denote the set of all continuous, nonnegative functions $f : [a,b] \to \mathbb{R}$. Prove that $CN[a,b]$ with the uniform convergence metric 
$$d(f, g) = \sup\{|f(x) - g(x)| : x \in [a,b]\}$$
is complete.

(b) Let $\lambda$ be an arbitrary constant. Prove that there is a unique nonnegative, continuous function $f : [0, 1] \to \mathbb{R}$ such that 
$$f(t) = 1 + \ln \left( 1 + \lambda \int_0^t f(s) \, ds \right), \quad 0 \leq t \leq 1.$$ 

Hint 1. For all $a \geq 0, b \geq 0$, $|\ln(1+a) - \ln(1+b)| \leq |a - b|$. You may use this inequality without proving it.

Hint 2. For a given $f \in CN[0,1]$, define 
$$\Theta(f)(t) = 1 + \ln \left( 1 + \lambda \int_0^t f(s) \, ds \right), \quad 0 \leq t \leq 1.$$ 
Prove that $\Theta(f) \in CN[0,1]$, and there is a positive integer $i$ such that $\Theta^i : CN[0,1] \to CN[0,1]$ is a contraction.