MATH 215 Homework 11

Turn in by May 9th, 2014 until 10:30 am.

Question 1. (20 × 2 = 40 points) Let \( f_n(x) = \frac{nx}{1+nx^2} \) and \( f(x) = \frac{1}{x} \).

(a) Prove that \( f_n \to f \) uniformly on \( E = [1, \infty) \).

(b) Prove that \( f_n \to f \) pointwise, but not uniformly on \( E = (0, 1) \).

Question 2. (20 points) Let \( f_n(x) = n^2xe^{-nx} \), and \( f(x) = 0 \). Prove that \( f_n \to f \) uniformly on every closed and bounded interval \( [a, b] \) where \( 0 < a < b \).

Question 3. (20 points) Let \( g : [0, 1] \to \mathbb{R} \) be a continuous function such that \( g(1) = 0 \). Let \( f_n(x) = g(x)x^n \), and \( f(x) = 0 \). Prove that \( f_n \to f \) uniformly on \( [0, 1] \).

Question 4. (20 points) Let \( C[0, 1] \) be the set of all continuous functions \( f : [0, 1] \to \mathbb{R} \) with the sup metric

\[
d(f, g) = \sup\{|f(s) - g(s)| : s \in [0, 1]\}.
\]

Prove that the closed ball \( B_1[0] \) in this space at center \( f = 0 \) (i.e., the constant function 0) and radius 1, is not compact.

*Hint.* Consider the sequence of functions \( \{f_n\} \in B_1[0] \), defined by \( f_n(s) = s^n \).