MATH 215 Homework 10

April 18, 2014

Turn in by April 30, 2014 until 10:30 am.

Question 1. (20 × 2 = 40 points)

Recall. For a fixed nonempty subset $A$ of a metric space $(X,d)$, we have defined $f_A : X \to \mathbb{R}$ by $f_A(x) = \text{dist}(x,A) = \text{g.l.b.}\{d(x,a) : a \in A\}$. We have also proved that $f_A$ is continuous on $X$.

(a) Prove that for an arbitrary point $p \in X$, $p \in \overline{A} \iff f_A(p) = 0$.

(b) Let $A$ and $B$ be two nonempty closed subsets of $X$, such that $A \cap B = \emptyset$. Prove that there are two nonempty open sets $E$ and $F$ in $X$, such that $A \subset E$, $B \subset F$, and $E \cap F = \emptyset$.

Hint. Consider the function $g : X \to \mathbb{R}$ defined by $g(x) = f_A(x) - f_B(x)$, and the sets $E = \{x \in X : g(x) < 0\}$, $F = \{x \in X : g(x) > 0\}$. Prove that the sets $E$ and $F$ have all the desired properties.

Remark. This fact is sometimes expressed by saying “in a metric space, two disjoint closed sets can be separated by two disjoint open sets”.

Question 2. (20 points)

Definition. Given a metric space $(X,d)$, and two points $p,q \in X$, a continuous function $\phi : [0,1] \to X$ (where the interval $[0,1] \subset \mathbb{R}$ has the absolute value metric), is called an arc, and it is said to join $p$ to $q$, if $\phi(0) = p$ and $\phi(1) = q$. An arc is said to lie in a subset $E$ of $X$, if $\phi(t) \in E$ for all $t \in [0,1]$. We say that a subset $E$ of $X$ is arcwise connected (or path connected), if for every pair of points $p,q \in E$, there is an arc joining $p$ to $q$ which lies in $E$.

Question. Let $(X,d)$ be an arbitrary metric space, and $E$ a nonempty subset of $X$. Prove that, $E$ is arcwise connected $\Rightarrow$ $E$ is connected.

Question 3. (20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a polynomial of degree 3, that is $f(x) = ax^3 + bx^2 + cx + d$, where $a,b,c,d$ are real constants, and $a \neq 0$. Prove that $f(\mathbb{R}) = \mathbb{R}$.

Question 4. (20 points) Let $X = B(\mathbb{N})$, i.e., the set of all bounded functions $f : \mathbb{N} \to \mathbb{R}$, with the sup metric. Note that a function $f : \mathbb{N} \to \mathbb{R}$ is a sequence $\{x_n\}$ in $\mathbb{R}$, where $x_n = f(n)$. So $X = B(\mathbb{N})$ is the set of all bounded sequences $\{x_n\}$ in $\mathbb{R}$, where for $p = \{x_n\} \in X$ and $q = \{y_n\} \in X$, we have that

$$d(p,q) = d(\{x_n\}, \{y_n\}) = \sup\{|x_n - y_n| : n \in \mathbb{N}\}.$$ 

Let $\emptyset$ denote the sequence $\emptyset = \{0,0,\ldots,0,\ldots\}$.

Let $p$ denote the sequence $\emptyset = \{0,0,\ldots,0,\ldots\}$.

Question. Prove that the closed ball $B_1[0]$ in $X$ is not compact.

Hint. Consider the sequence $\{e^k\}$ in $X$ defined by

$$e^1 = \{1,0,0,\ldots\}, e^2 = \{0,1,0,0,\ldots\}, e^3 = \{0,0,1,0,0,\ldots\}, \ldots e^k = \{0,\ldots,0,1,\ldots\}, \ldots$$

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