MATH 215 MIDTERM 1

Q 1. A real number $x$ which is not a rational number is called an **irrational number**.

(a) Prove that the sum of two rational numbers is rational.
(b) Prove that the sum of an irrational number and a rational number is irrational.

Solution.
(a) Let $a$ and $b$ be two rational numbers. Then there are integers $m, n, p, q$ such that $n \neq 0, q \neq 0,$ and
\[
a = \frac{m}{n}, \quad b = \frac{p}{q}.
\]
Then
\[
a + b = \frac{m}{n} + \frac{p}{q} = \frac{mq + np}{nq}.
\]
Since $mq + np$ and $nq$ are integers, and $nq \neq 0$, we have that $a + b$ is also a rational number.

(b) Let $a$ be a rational number, and $x$ be an irrational number. First note that $-a$ is also a rational number since $-a = \frac{-m}{n}$, and $-m, n$ are both integers. Now assume to the contrary that $b = a + x$ is a rational number. Then $x = b - a = b + (-a)$ would be a rational number by part (a). A contradiction.

Q 2. Let $E$ be a nonempty subset of $\mathbb{R}$ which is bounded above. Prove that $\sup E$ belongs to the closure of $E$, that is $\sup E \in \overline{E}$.

Proof. By the Least Upper Bound Property of $\mathbb{R}$, we know that $\alpha = \sup B$ exists in $\mathbb{R}$. Let $N$ be any neighborhood of $\alpha$. We must show that $N \cap E \neq \emptyset$. By definition of neighborhood, there is a number $r > 0$ such that $B_r(\alpha) \subseteq N$. Note that in $\mathbb{R}$, $B_r(\alpha) = (\alpha - r, \alpha + r)$. Since $\alpha - r < \alpha$, i.e., $\alpha - r$ is smaller than the smallest upper bound of $E$, $\alpha - r$ is not an upper bound for $E$. So there is an element $x \in E$ such that $\alpha - r < x$. We also have that $x \leq \alpha < \alpha + r$. Thus
\[
\alpha - r < x < \alpha + r \Rightarrow x \in (\alpha - r, \alpha + r) = B_r(\alpha).
\]
So
\[
x \in B_r(\alpha) \cap E \subseteq N \cap E \Rightarrow N \cap E \neq \emptyset.
\]

Q 3. Let $\mathcal{S} = \{A_1, A_2, A_3, \ldots, A_n, \ldots\}$ be a countable set of sets $A_1, A_2, \ldots, A_n, \ldots$. Assume that all of the sets $A_1, A_2, \ldots, A_n, \ldots$ are countable. Prove that their union $\bigcup_{n=1}^{\infty} A_n$ is also countable.

Proof. Since each set $A_n$ is countable, we write its elements as a sequence
\[
A_1 = \{a_1^1, a_1^2, a_1^3, a_1^4, \ldots\}
\]
\[
A_2 = \{a_2^1, a_2^2, a_2^3, a_2^4, \ldots\}
\]
\[ A_3 = \{a_1^3, a_2^3, a_3^3, a_4^3, \ldots\} \]
\[ A_4 = \{a_1^4, a_2^4, a_3^4, a_4^4, \ldots\} \]
\[ \vdots \]

Then we write these elements in an array

<table>
<thead>
<tr>
<th>1st</th>
<th>2nd</th>
<th>3rd</th>
<th>4th</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_1 )</td>
<td>( a_1^1 )</td>
<td>( a_2^1 )</td>
<td>( a_3^1 )</td>
<td>( a_4^1 )</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>( a_1^2 )</td>
<td>( a_2^2 )</td>
<td>( a_3^2 )</td>
<td>( a_4^2 )</td>
</tr>
<tr>
<td>( A_3 )</td>
<td>( a_1^3 )</td>
<td>( a_2^3 )</td>
<td>( a_3^3 )</td>
<td>( a_4^3 )</td>
</tr>
<tr>
<td>( A_4 )</td>
<td>( a_1^4 )</td>
<td>( a_2^4 )</td>
<td>( a_3^4 )</td>
<td>( a_4^4 )</td>
</tr>
</tbody>
</table>
| \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | \( \vdots \) | ...

Then we list the elements of the set \( \bigcup_{n=1}^{\infty} A_n \) as a sequence by using the Cantor’s counting scheme as indicated by the arrows in the above array:

\[ a_1^1, a_2^1, a_1^2, a_2^2, a_1^3, a_2^3, a_1^4, a_2^4, a_3^1, a_3^2, a_3^3, a_3^4, \ldots \]

Thus \( \bigcup_{n=1}^{\infty} A_n \) is countable.

**Q 4.** Let \( (X, d) \) be a metric space.

(a) Prove that if \( E \) and \( F \) are two subsets of \( X \) with \( E \subseteq F \), then \( E^o \subseteq F^o \).

(b) Prove that if \( A \) and \( B \) are two subsets of \( X \), then \( (A \cap B)^o = A^o \cap B^o \).

**Proof.**

(a) Let \( p \in E^o \). Then there is a number \( r > 0 \) such that \( B_r(p) \subseteq E \). Since \( E \subseteq F \), we have that \( B_r(p) \subseteq F \). But this is exactly saying \( p \in F^o \).

(b)

\[
A \cap B \subseteq A \Rightarrow (A \cap B)^o \subseteq A^o \text{ by (a)}
\]
\[
A \cap B \subseteq B \Rightarrow (A \cap B)^o \subseteq B^o \text{ by (a)}
\]

Thus \( (A \cap B)^o \subseteq A^o \cap B^o \).

Conversely let \( p \in A^o \cap B^o \).

\[
p \in A^o \Rightarrow \exists r_A > 0 \text{ s.t. } B_{r_A}(p) \subseteq A,
\]
\[
p \in B^o \Rightarrow \exists r_B > 0 \text{ s.t. } B_{r_B}(p) \subseteq B.
\]

Let \( r = \min\{r_A, r_B\} \). Then \( r > 0 \), and

\[
r \leq r_A \Rightarrow B_r(p) \subseteq B_{r_A}(p) \Rightarrow B_r(p) \subseteq A,
\]
\[
r \leq r_B \Rightarrow B_r(p) \subseteq B_{r_B}(p) \Rightarrow B_r(p) \subseteq B.
\]

Thus \( B_r(p) \subseteq A \cap B \), which means \( p \in (A \cap B)^o \).
Q 5. For the following sets $E$, find $E^o, E', \overline{E}$. No proofs are needed.

(a) $X = \mathbb{R}$, $d(x, y) = |x - y|$, $E = \mathbb{Q}$.

Solution. $E^o = \emptyset$, $E' = \mathbb{R}$, $\overline{E} = \mathbb{R}$.

(b) $X = \mathbb{R}^2$, $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$, 
$E = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 \leq 9\} \cup \{(x, y) \in \mathbb{R}^2 : -3 \leq x \leq 3, y = 2\}$.

In part (b) draw the picture of $E$.

Solution. First we draw a picture of $E$.

We have

\[
\begin{align*}
E^o &= \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 9\}, \\
E' &= \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 9\} \cup \{(x, y) \in \mathbb{R}^2 : -3 \leq x \leq 3, y = 2\}, \\
\overline{E} &= \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 9\} \cup \{(x, y) \in \mathbb{R}^2 : -3 \leq x \leq 3, y = 2\}.
\end{align*}
\]