Date: 18 June 2005

MATH 215 MIDTERM 1 SOLUTIONS

Q 1. Let $A$ and $B$ be two nonempty subsets of $\mathbb{R}$ that are bounded below. Assume $\inf A < \inf B$. Show that there is an element $a_0 \in A$ such that $a_0$ is a lower bound for the set $B$.

Solution. Let $b^* = \inf B$. Since $\inf A < b^*$, i.e., the number $b^*$ is strictly greater than the greatest lower bound of the set $A$, the number $b^*$ cannot be a lower bound for the set $A$. So there is an element $a_0 \in A$ such that $a_0 < b^*$. If $x$ is an arbitrary element of $B$, then $b^* \leq x$ (since $b^*$ is a lower bound for the set $B$). Then by the trichotomy law we have that $a_0 < x$. So $a_0$ is less than all elements of $B$, i.e., $a_0$ is a lower bound for $B$.

Q 2. Let $\tilde{C}$ denote the set of all complex numbers $x + iy$ where both $x$ and $y$ are rational numbers. Show that the set $\tilde{C}$ countable.

Solution. We use Cantor’s counting scheme. We know that the set of all rational numbers $\mathbb{Q}$ is countable. So we can write the elements of $\mathbb{Q}$ as a sequence.

$$\mathbb{Q} = \{q_1, q_2, q_3, q_4, \ldots\}.$$  

We consider the following infinite array of the elements of $\tilde{C}$.

<table>
<thead>
<tr>
<th></th>
<th>$q_1$</th>
<th>$q_2$</th>
<th>$q_3$</th>
<th>$q_4$</th>
<th>$\ldots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q_1$</td>
<td>$q_1 + iq_1$</td>
<td>$q_1 + iq_2$</td>
<td>$q_1 + iq_3$</td>
<td>$q_1 + iq_4$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$q_2$</td>
<td>$q_2 + iq_1$</td>
<td>$q_2 + iq_2$</td>
<td>$q_2 + iq_3$</td>
<td>$q_2 + iq_4$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$q_3$</td>
<td>$q_3 + iq_1$</td>
<td>$q_3 + iq_2$</td>
<td>$q_3 + iq_3$</td>
<td>$q_3 + iq_4$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$q_4$</td>
<td>$q_4 + iq_1$</td>
<td>$q_4 + iq_2$</td>
<td>$q_4 + iq_3$</td>
<td>$q_4 + iq_4$</td>
<td>$\ldots$</td>
</tr>
</tbody>
</table>

Then we define $F : \mathbb{N} \rightarrow \tilde{C}$ as follows

$$\begin{array}{cccccccc}
\mathbb{N} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots \\
\tilde{C} & q_1 + iq_1 & q_1 + iq_2 & q_2 + iq_1 & q_1 + iq_3 & q_2 + iq_2 & q_3 + iq_1 & q_1 + iq_4 & q_2 + iq_3 & \ldots \\
\end{array}$$

This way we get a 1-1 correspondence between $\mathbb{N}$ and $\tilde{C}$. 
Q 3. Let $X$ and $Y$ be two nonempty sets and $f : X \to Y$ be a function. Show that for all subsets $B_1 \subset Y$ and $B_2 \subset Y$ we have

$$f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2).$$

Solution. Let $x \in f^{-1}(B_1 \cap B_2)$. Then by definition of the inverse image $f(x) \in B_1 \cap B_2$. So $f(x) \in B_1$ and $f(x) \in B_2$. It means that $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$, i.e., $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. So $f^{-1}(B_1 \cap B_2) \subset f^{-1}(B_1) \cap f^{-1}(B_2)$.

Conversely let $x \in f^{-1}(B_1) \cap f^{-1}(B_2)$. Then $x \in f^{-1}(B_1)$ and $x \in f^{-1}(B_2)$, so $f(x) \in B_1$ and $f(x) \in B_2$. Thus $f(x) \in B_1 \cap B_2$, which means by the definition of the inverse images that $x \in f^{-1}(B_1 \cap B_2)$. Thus $f^{-1}(B_1 \cap B_2) \subset f^{-1}(B_1 \cap B_2)$.

Q 4. Let $E$ and $F$ be two nonempty subsets of a metric space. Show that

$$E \cup F = \overline{E} \cup \overline{F}.$$  

Solution. We are going to use the following:

(a) For any subset $A$ of $X$, $\overline{A}$ is a closed set.
(b) For any subset $A$ of $X$, $A \subset \overline{A}$.
(c) If $A \subset B$ and $B$ is closed, then $\overline{A} \subset B$.
(d) The union of finitely many closed sets is closed, hence in particular the union of two closed sets is closed.

We have

$$E \subset E \cup F \overset{(b)}{\subset} \overline{E \cup F} \overset{(a)}{\supset} \overline{E} \overset{(c)}{\subset} E \subset E \cup F.$$  

Similarly $F \subset \overline{E \cup F}$. So taking the union we get $E \cup F \subset \overline{E \cup F}$.

Conversely, $E \overset{(b)}{\subset} \overline{E}$ and $F \overset{(b)}{\subset} \overline{F}$ implies $E \cup F \subset \overline{E \cup F}$.

By (a) and (d), the set $E \cup F$ is closed. By (c), we have that $E \cup F \subset \overline{E \cup F}$.

Q 5. For the following sets $E$, draw the picture of $E$, find $\text{int} E, E', \overline{E}$. Determine whether the sets $E$ are connected and compact. No proofs are necessary.

a) $X = \mathbb{R}^2$, $E = \{(x, y) : 1 < x^2 + y^2 \leq 4\} \cup \{(0, 0)\}$.

Solution.

$$\text{int} E = \{(x, y) : 1 < x^2 + y^2 < 4\}$$

$$E' = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$$

$$\overline{E} = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\} \cup \{(0, 0)\}.$$  

The set $E$ is not connected and not compact.
b) \( X = \mathbb{R}^2, E = \{(\frac{1}{m}, \frac{1}{n}) : m, n \in \mathbb{N}\}. \)

Solution.

\[
\text{int}E = \emptyset
\]

\[
E' = \left\{(1, 0), \left(\frac{1}{2}, 0\right), \left(\frac{1}{3}, 0\right), \ldots, (0, 1), \left(0, \frac{1}{2}\right), \left(0, \frac{1}{3}\right), \ldots, (0, 0)\right\}
\]

\[
E = \left\{\left(\frac{1}{m}, \frac{1}{n}\right) : m, n \in \mathbb{N}\right\} \cup \left\{(1, 0), \left(\frac{1}{2}, 0\right), \left(\frac{1}{3}, 0\right), \ldots, (0, 1), \left(0, \frac{1}{2}\right), \left(0, \frac{1}{3}\right), \ldots, (0, 0)\right\}
\]

The set \( E \) is not connected and not compact.