(20 points) 1. The region between the graphs of \( y = x^2 \) and \( y = \frac{1}{2} - x^2 \) is revolved about the \( y \)-axis to form a lens. Compute the volume of the lens.

(20 points) 2. By using definition show that

\[
\lim_{x \to 1} \left( x^2 + \frac{1}{x} \right) = 2.
\]

(7+8+5 points) 3. Let \( f : [a, b] \to \mathbb{R} \) be a function such that

\[
\forall x \in [a, b] \quad a \leq f(x) \leq b
\]

and

\[
\forall u, v \in [a, b] \text{ with } u \neq v, \quad |f(u) - f(v)| < |u - v|.
\]

a) Show that \( f \) is continuous on \([a, b]\).

b) Show that there is at least one point \( c \in [a, b] \) such that \( f(c) = c \).

c) Show that there is only one \( c \in [a, b] \) such that \( f(c) = c \).

(20 points) 4. Let \( I \) be an open interval containing the point \( c \). Suppose \( f \) is continuous on \( I \), and \( f'(x) \) exists for all \( x \in I \), except perhaps for \( x = c \). Suppose moreover that \( \lim_{x \to c} f'(x) = L \) exists. Show that \( f'(c) \) also exists and \( f'(c) = L \).

(20 points) 5. The graph in the figure is the graph of the derivative \( y' = f'(x) \) of some function \( f \). Suppose \( f(0) = 0 \). Sketch the graph of \( y = f(x) \). On your graph mark all local extrema and points of inflection.

(10+10 points) 6.

a) Evaluate

\[
\int \frac{x^{39}}{(x^{10} + 1)^5} \, dx
\]

b) Let \( n \geq 2 \) be an integer. By using the method of integration by parts derive the reduction formula

\[
\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx.
\]
(15 points) 1. By using only definition (i.e. $\varepsilon$ and $\delta$), show that
\[
\lim_{x \to 1} \frac{x^3 + 2x^2 - x - 2}{x - 1} = 6.
\]

(15 points) 2. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and assume that $f'$ is a bounded function on $\mathbb{R}$. Show that $f$ is uniformly continuous on $\mathbb{R}$.

(15+5 points) 3. 
(a) Suppose $f(a) = g(a) = h(a)$ and $f(x) \leq g(x) \leq h(x)$ for all $x$ in a neighborhood $N(a)$ of $a$. Also suppose $f'(a) = h'(a)$. Show that $g$ is differentiable at $a$, and $g'(a) = f'(a) = h'(a)$.
(b) Give an example to show that the conclusion does not hold if we remove the hypothesis $f(a) = g(a) = h(a)$.

(20 points) 4. Let $f(x) = \log \left| \frac{1}{1-x} \right|$. Find the domain, intercepts, asymptotes. Examine the sign of the first and second derivatives. Sketch the graph of the function. On your graph mark all local extrema and points of inflection if there are any.

(15 points) 5. Let $\alpha > 1$ be a rational number. By using maximum-minimum methods, show that
\[
\forall x \geq -1, \quad (1 + x)^\alpha \leq 1 + \alpha x.
\]

(9+9+9+9 points) 6. 
(a) Let $f(x) = g(x^3) h(x^2 + 3x)$, where $g$ and $h$ are differentiable functions. Given that
\[
\begin{align*}
h(10) &= 3, & h'(10) &= -1, \\
g(24) &= 5, & g'(24) &= 4,
\end{align*}
\]

find $f'(2)$.
(b) Evaluate $\int \frac{\sin^3 x}{\cos^2 x} \, dx$
(c) Find $\frac{dy}{dx}$ where $y = f(x)$ is implicitly defined by
\[
x = \int_0^y \frac{t^2 \, dt}{\sqrt{1 + t^4}}.
\]
(d) Find a reduction formula for
\[
I_n = \int \sec^n x \, dx
\]
where $n \geq 3$ is an integer.
1) A function $f$ is defined as follows:

$$f(x) = \begin{cases} 
  x^2 & \text{if } x \leq c, \\
  ax + b & \text{if } x > c,
\end{cases}$$

where $a, b, c$ are constants. Find the values of $a$ and $b$ in terms of $c$ such that $f'(c)$ exists.

2) Let $f$ be a continuous function on $[0, 1]$ such that $0 \leq f(x) \leq 1$ for all $x$ in $[0, 1]$. Show that there is a $c$ in $[0, 1]$ for which $f(c) = c$.

3) Show that $\lim_{x \to 0} x(-1)^{1/x} = 0$.

4) Compute $f'$ if $f$ is defined by:

$$f(x) = \begin{cases} 
  x^2 \sin(1/x) & \text{if } x \neq 0, \\
  0 & \text{if } x = 0.
\end{cases}$$

5) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f(q) = 0$ for all $q \in \mathbb{Q}$. Show that if $f$ is continuous at $\sqrt{2}$, then $f(\sqrt{2}) = 0$.

6) Let $f$ be an increasing function on $[a, b]$. Show that $\lim_{x \to p^+} f(x)$ exists for all $x \in [a, b]$.

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>Total</th>
</tr>
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<td>8</td>
<td>8</td>
<td>10</td>
<td><strong>50</strong></td>
</tr>
</tbody>
</table>
MATH 113 MIDTERM EXAM II

NOTES: Write your solutions in a readable form. A good proof is recognized by the scarcity of symbols and by the abundance of explanations in it. Grading is always influenced by good exposition.

QUESTIONS:

1) Show that the lateral surface area of a cone of a fixed volume is minimized when the height of the cone is \( \sqrt{2} \) times the radius of the base.

2) Assume that there are walls in the plane along the lines \( \{ x = 0, y \geq 0 \} \), \( \{ y = 0, x \geq 0 \} \), \( \{ x = a, y \geq b \} \) and \( \{ y = b, x \geq a \} \), where \( a \) and \( b \) are positive real numbers. Find the length of the longest beam that can be carried around the corner.

3) Evaluate the following integrals:

\[
\int \frac{\sin \ln x}{x^2} \, dx, \quad \int \sqrt{\tan x} \sec^4 x \, dx, \quad \int e^{ax} \cos bx \, dx \text{ where } a, b \in \mathbb{R}
\]

4) Assume that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is differentiable at 0 and that \( f(0) = 0 \). Prove that there exists a continuous function \( g(x) \) such that \( f(x) = xg(x) \).

Every question is 25 points.
Math 113 Exam 2

Date: 16 December 1995

1. Evaluate the following limits
   a) \( \lim_{x \to 1} \frac{1 - \sqrt{x}}{x - 1} \)
   b) \( \lim_{x \to 0} \frac{x^2(3 + \sin x)}{(x + \sin x)^2} \)

2. Let \( f(x) = \sqrt{x}, \ x \geq 0 \). Show that \( f \) is uniformly continuous on the set \( S = [1, +\infty) \).

3. For both parts, assume \( f : [a, b] \to \mathbb{R} \) is differentiable on the closed interval \( [a, b] \), but do not assume that \( f' \) is continuous.
   a) Show that if the minimum of \( f \) on \( [a, b] \) is at \( a \) then \( f'(a) \geq 0 \), and if the minimum of \( f \) on \( [a, b] \) is at \( b \) then \( f'(b) \leq 0 \).
   b) Suppose that \( f'(a) < 0 \) and \( f'(b) > 0 \). Show that \( f'(c) = 0 \) for some \( c \) in the open interval \( (a, b) \). *Hint:* Consider the minimum of \( f \) on \( [a, b] \); why must it be in the open interval \( (a, b) \)?

4. A straight line is drawn from the point \((0, a)\) to the horizontal axis and then to \((1, b)\) as in the figure. For which \( x \) is the total length shortest?

5. Evaluate the following integrals
   a) \( \int \sin \sqrt{x} + 1 \, dx \)
   b) \( \int_0^1 \frac{2x + 3}{(6x + 7)^3} \, dx \)
MATH 113 MIDTERM EXAM II

1. Without using any theorems about uniform continuity, show that the function \( f(x) = x^2 + \frac{1}{x} \) is uniformly continuous on the closed interval \( S = [1, 3] \). \[20 \text{ points}\]

2. Let \( f \) be a function defined for all \( x > 0 \) and assume that \( \lim_{x \to 0^+} \frac{f(x)}{3x^4 + 2x^3} = 5 \). Let

\[
g(x) = \begin{cases} 
\frac{f(x)}{x^2} & \text{if } x > 0 \\
0 & \text{if } x = 0 \\
\frac{f(-x)}{x^2} & \text{if } x < 0
\end{cases}
\]

Show that \( g \) is differentiable at \( x = 0 \) and find \( g'(0) \). \[15 \text{ points}\]

3. Find the height of the trapezoid of largest area that can be inscribed in a semicircle of radius \( c \), the lower base being on the diameter. \[15 \text{ points}\]

4. \[a) \] Let \( f \) and \( g \) be differentiable functions such that \( f' \) and \( g' \) are continuous. Assume that for all \( t \geq 0 \), \( f'(t) \leq g'(t) \) and \( f(0) = g(0) \). Show that for all \( x \geq 0 \), \( f(x) \leq g(x) \).

*Hint.* Use Fundamental Theorem of Calculus.

\[b) \] Let \( f \) be a differentiable function such that \( f' \) is continuous. Assume also that \( f(0) = 0 \) and \( 0 < f'(x) \leq 1 \) for all \( x \geq 0 \). Show that for all \( x \geq 0 \)

\[
f(x)^2 \leq 2 \int_0^x f(t) \, dt.
\]

5. Evaluate the following integrals. You may not use any reduction formula or any integral evaluated in the class or in the book.

\[a) \int \ln(x + \sqrt{x}) \, dx \quad b) \int \sqrt{x^2 - 1} \, dx \quad c) \int \sqrt{1 + e^x} \, dx \]

[10+10+10 points]
1. a) Evaluate the following limit

\[ \lim_{x \to \frac{\pi}{2}} \frac{(x - \frac{\pi}{2}) \cos x}{1 - \sin x}. \]

b) Let

\[ \lim_{x \to -2} \frac{x^3 - Ax + 2}{x + 2} = B \]

where \( A \) and \( B \) are constants which are real numbers. Find the values of \( A \) and \( B \).

2. Let \( f : [0, 1] \to \mathbb{R} \) be continuous at every point of \([0, 1]\) and assume for every \( x \) with \( 0 \leq x \leq 1 \) we have \( 0 \leq f(x) \leq 1 \).

a) Show that the graph of \( f \) intersects the line \( y = 1 - x \), i.e., there is at least one number \( c \) such that \( 0 \leq c \leq 1 \) and \( f(c) = 1 - c \).

b) Furthermore, assume that \( f'(x) \) exists and \( f'(x) > -1 \) for every \( x \) with \( 0 < x < 1 \). Show that in this case there is at most one \( c \in (0, 1) \) such that \( f(c) = 1 - c \).

3. Let \( A \) and \( B \) be real constants and \( f \) be defined by

\[ f(x) = \begin{cases} 
Ax + 1 & \text{if } x \leq 0 \\ 
B + 2x + x^2 \sin \frac{1}{x} & \text{if } 0 < x 
\end{cases} \]

Find the values of the constants \( A \) and \( B \) for which \( f \) is differentiable at \( x = 0 \). Is \( f' \) continuous at \( x = 0 \)?

4. The point \((u, v)\) is on the ellipse \( x^2 + 2y^2 = 2 \) and the line segment \( PQ \) is tangent to the ellipse at the point \((u, v)\) (where \( P \) is on the positive \( y \)-axis, \( Q \) is on the positive \( x \)-axis.)

a) Find the coordinates of \( P \) and \( Q \) in terms of \( u \) and \( v \).

b) For which value of \((u, v)\) is the area of the triangle \( POQ \) smallest?

5. a) Evaluate: \( \int \frac{x^2}{(2x + 1)^{10}} \, dx \).

b) Let \( n \) be a positive integer. Find a reduction formula for the integral

\[ \int \cos^n x \, dx. \]
MATH 113 MIDTERM EXAM II

1. Let \( f : \mathbb{R} \to \mathbb{R} \) be a function such that
   \[ f(u + v) = f(u)f(v) \quad \text{for all } u, v \in \mathbb{R}. \]
   Assume also that \( f \) is not a constant function.
   a) Show that \( f(0) = 1 \).
   b) Given that \( f'(0) = 3 \), find \( f'(x) \) in terms of \( f(x) \).

2. Evaluate the following derivatives.
   a) Find \( y' \) if \( y = \sin((\cos x)^2) \).
   b) Find \( y' \) if \( y = x\sqrt{1 + x^2} \).
   c) Find \( y' \) at the point \((x, y) = (\frac{\pi}{2}, 1)\) if \( \sin(xy) + 2y^2 = 3 \).

3. Find the equation(s) of the tangent line(s) from the point \( P_0 = (0, -4) \) to the curve \( y = x^2 - 3x \). (Note that \( P_0 \) is not on the curve).

4. A variable right triangle \( ABC \) in the \( xy \)-plane has a fixed vertex \( A \) at the origin, a vertex \( C \) restricted to lie on the parabola \( y = 1 + 3x^2, x > 0 \), and has right angle at the vertex \( B \) on the \( y \)-axis. The point \( B \) starts at the point \((0, 1)\) at time \( t = 0 \) and moves upward along the \( y \)-axis at a constant velocity of \( 2 \text{cm/sec} \). How fast is the area of the triangle changing when \( t = \frac{9}{2} \text{sec} \)?

5. Sketch the graph of \( f(x) = x^4 - 4x^3 \) by considering \( f'(x) \) and \( f''(x) \). On your graph mark all extrema and inflection points.

6. Let \( k \) be a real constant such that
   \[ k \leq 27x^2 + \frac{8}{x} \quad \text{for all real numbers } x > 0. \]
   Show that \( k \leq 18\sqrt{2} \).