MATH 525 MIDTERM II

1. Let $\mathbb{Z}_m$ be ring of residues modulo $m$ and $\mathbb{Z}_m^*$ be its multiplicative group consisting of residue classes of integers coprime to $m$. Consider Gaussian sum
   \[ G_m(\psi) = \sum_{x \in \mathbb{Z}_m} \psi(x)e^{2\pi ix/m}, \]
   where $\psi : \mathbb{Z}_m^* \to \mathbb{C}^*$ is a character of the multiplicative group extended by zeros to a complex function on $\mathbb{Z}_m$. Make use of character theory to prove that $|G_m(\psi)| = \sqrt{m}$.

2. Let $f(x) = \sum_k a_k \chi_k(x)$ be Fourier decomposition over irreducible characters $\chi_k$ of a central function $f : G \to \mathbb{C}$ on finite group $G$. Prove the following Plancherel formula
   \[ \frac{1}{|G|} \sum_{x \in G} |f(x)|^2 = \sum_k |a_k|^2. \]

3. Prove that finite abelian group $G$ is determined by its character table uniquely up to isomorphism. Does a similar result hold for nonabelian groups?

4. Let $G$ be a finite abelian group and $\hat{G}$ be its group of characters $\chi : G \to \mathbb{C}^*$ called dual abelian group. Prove the following properties of the duality: (i) $|G| = |\hat{G}|$; (ii) $\sigma : G \to \hat{G}$, $x \mapsto (\chi \mapsto \chi(x))$ is an isomorphism; (iii) $\hat{G} \times H = \hat{G} \times \hat{H}$; (iv) $G$ is isomorphic to its dual group $\hat{G}$.

Let’s now return to the settings of Problem 10 MTI and denote by $V$ the space of functions on facets of a cube. Recall that it splits into three irreducible components $V = U_1 \oplus U_2 \oplus U_3$ with respect to rotation group $G \simeq S_4$ of the cube.

5. Find $G$-invariant projectors $E_i$ onto the irreducible components $U_i$.

6. For function $f \in V$ write down explicitly its decomposition $f = f_1 + f_2 + f_3$, where $f_i \in U_i$

7. and deduce an explicit description of the components $U_i$, e.g. $U_1$ consists of constant functions etc.

Consider now linear operator $T : V \to V$ that change a value of a function $f(x)$ on facets $x$ to its mean value on the adjacent facets $Tf(x) = \frac{1}{4} \sum_{y \text{adj}} f(y)$.

8. Show that $T$ commutes with action $G : V$

9. and deduce that $T$ acts as a scalar $\lambda_i$ on irreducible component $U_i$.

10. Find the scalars $\lambda_i$.