SCBA for equilibrium and transport

- $B = 0$ selfenergy of the 2DES
- Finite-$B$ selfenergy of the 2DES
- Exact formulas for the conductivity tensor
- $B = 0$ conductivities
- Magnetoconductivities
For $B = 0$, the energy eigenvalues and -functions of the homogeneous 2D ES are

$$\varepsilon(k) = \frac{\hbar^2 k^2}{2m} \quad \text{and} \quad \varphi_k(r) = \langle r|k \rangle = \frac{1}{\sqrt{\mathcal{F}}} e^{i k \cdot r}.$$ 

It is useful to represent the impurity potentials $v_R(r) \equiv v(r - R)$ by their Fourier coefficients, defined by

$$v_q = \int d^2 r \ v(r) e^{-i q \cdot r}, \quad v(r) = \frac{1}{\mathcal{F}} \sum_q e^{i q \cdot r} v_q \rightarrow \int \frac{d^2 q}{(2\pi)^2} e^{i q \cdot r} v_q.$$ 

Then the matrix elements of the impurity potentials (here $\hat{r}$ is an operator) are

$$\langle k' | v_R(\hat{r}) | k \rangle = \frac{1}{\mathcal{F}} e^{i(k-k') \cdot R} v_{k' - k},$$

and the evaluation of the selfenergy yields

$$\langle k' | \hat{\Sigma}(z) | p' \rangle = \sum_{k,p} W_{k,p}^{k',p'} \langle k | \hat{G}(z) | p \rangle \quad \text{with} \quad W_{k,p}^{k',p'} = \delta_{k' - p', k - p} \frac{n_f}{\mathcal{F}} |v_{k' - k}|^2.$$
Diagonality of the Green function

The unitary translation operator \( \hat{T}_a = \exp(ia \cdot \hat{p}/\hbar) = \exp(a \cdot \nabla) \) shifts the position by the vector \( a \): for arbitrary \( \langle r | \psi \rangle = \psi(r) \) one has \( \langle r | \hat{T}_a | \psi \rangle = \psi(r + a) \) and \( \langle r | \hat{T}_a \nu(\hat{r} - R) | \psi \rangle = \nu(r + a - R)\psi(r + a) \), which implies \( \hat{T}_a \nu_R(\hat{r}) = \nu_{R-a}(\hat{r}) \hat{T}_a \), and, since \( \hat{T}_a^\dagger = \hat{T}_a^{-1} = \hat{T}_{-a} \), \( \hat{T}_a \nu_R(\hat{r}) \hat{T}_a^\dagger = \nu_{R-a}(\hat{r}) \). Since \( \hat{T}_a \) commutes with \( H_0 = \hat{p}^2/2m \):

\[
\langle r | \hat{G}(z) | r' \rangle = \left( \prod_{j=1}^{N} \int \frac{d^2R_j}{\mathcal{F}} \right) \langle r | \hat{T}_a^\dagger \hat{T}_a \left( z - H_0 - \sum_{j=1}^{N} \nu_{R_j}(\hat{r}) \right)^{-1} \hat{T}_a^\dagger \hat{T}_a | r' \rangle
\]

\[
= \left( \prod_{j=1}^{N} \int \frac{d^2R_j}{\mathcal{F}} \right) \langle r + a | \left( z - H_0 - \sum_{j=1}^{N} \nu_{R_j-a}(\hat{r}) \right)^{-1} | r' + a \rangle.
\]

We assume an on the average homogeneous, periodically repeated impurity distribution and shift the range of all \( R_j \)-integrals by \( a \), i.e., replace in the last line the subscripts \( R_j-a \) by \( R_j \). Thus, we see that \( \langle r | \hat{G}(z) | r' \rangle = \langle r + a | \hat{G}(z) | r' + a \rangle \) for arbitrary values of \( a \). With the choice \( a = -(r + r')/2 \), we find \( \langle r | \hat{G}(z) | r' \rangle = \langle (r-r')/2 | \hat{G}(z) | -(r-r')/2 \rangle \). Then \( \hat{G} \) is diagonal in the momentum representation:

\[
\langle k | \hat{G}(z) | p \rangle = \mathcal{F}^{-1} \int d^2r \int d^2r' e^{-ik \cdot r} \langle (r-r')/2 | \hat{G}(z) | -(r-r')/2 \rangle e^{ip \cdot r'}
\]

\[
= \delta_{k,p} \int d^2\tilde{r} \langle \tilde{r}/2 | \hat{G}(z) | -\tilde{r}/2 \rangle e^{ip \cdot \tilde{r}}.
\]
Diagonality and symmetry of the selfenergy

since

$$\langle k' | \hat{\Sigma}(z) | p' \rangle = \sum_{k,p} W_{k,p}^{k',p'} \langle k | \hat{G}(z) | p \rangle \quad \text{with} \quad W_{k,p}^{k',p'} = \delta_{k'-p'+k-p} \frac{n_i}{J} | v_{k'} - k |^2 ,$$

the diagonality of $\hat{G}$ implies that of $\hat{\Sigma}$:

$$\langle k' | \hat{G}(z) | k \rangle = \delta_{k',k} G_k(z) , \quad \Rightarrow \quad \langle k' | \hat{\Sigma}(z) | k \rangle = \delta_{k',k} \Sigma_k(z) .$$

We assume rotational invariance of the impurity potentials, $v_k = v_k$, then the averaged 2D ES is isotropic and $G_k(z)$ and $\Sigma_k(z)$ depend only on $k = |k|$, that is on the energy $\varepsilon_k = \varepsilon(k)$. The SCBA relation between selfenergy and Green function reduces to

$$\Sigma_k(z) = n_i \int \frac{d^2 k'}{(2\pi)^2} v_{|k-k'|}^2 G_{k'}(z) = \Gamma_0 \int_0^\infty d\varepsilon' \int_{-\pi}^\pi \frac{d\chi}{2\pi} w(k, k'; \chi) G_{k'}(z) \bigg|_{k' = \sqrt{2m\varepsilon'/\hbar^2}} ,$$

where $\Gamma_0 = n_i v_0^2 m/(2\pi\hbar^2)$ and

$$w(k, k'; \chi) = \left[ v_q/v_0 \right]^2 \quad \text{with} \quad q = \sqrt{k^2 + k'^2 - 2kk' \cos \chi}$$

is a dimensionless scattering cross section, depending on shape and range of the impurity potential. With $G_k(z) = 1/[z - \varepsilon_k - \Sigma_k(z)]$: non-linear integral equation for $\Sigma_k$. 

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Analytical properties of Green’s functions

- The integral equation for $\Sigma_k$ depends on the parameter $z$.
- If the impurity potential has finite range, its Fourier components will assure convergence of the $k'$-integral.
- $\Sigma_k(z)$ and $G_k(z)$ should be analytic in the upper complex half-plane $\text{Im}z > 0$.
- For spectral function and DOS we need $G_k(z)$ only near the real axis of the complex $z$-plane, $G_k^\pm(E) = G_k(E \pm i0)$.
- If $G_k(z)$ is analytic in $\text{Im}z > 0$, its contour integral over any closed contour in the upper half-plane vanishes.

Consider semi-circle $z = R e^{i\varphi}$ for $0 < \varphi < \pi$ and the straight line $z = E + i0$ for $-R < E < R$. If for $R \to \infty$ the ratio $\Sigma_k(z)/z \to 0$ vanishes on the semi-circle, the contour integral along the semi-circle is easily calculated, $\int_{sc} dz \, G_k(z) \to i\pi$. Since $\text{Im}G_k(E + i0) = -\pi A_k(E)$,

$$\int_{-\infty}^{\infty} dE \, A_k(E) = 1.$$ 

Therefore, one should only consider approximations which do not violate these analyticity requirements.
A toy model ("impurity jelly")

We first consider the (unrealistic) limit in which the range of the impurity potential increases indefinitely while its strength decreases in such a manner, that the Fourier coefficients for $|q| \to 0$ survive.

$$n_I v^2(q)/(2\pi)^2 = \Gamma_j^2 \delta(q) \quad \Rightarrow \quad \Sigma_k(z) = \Gamma_j^2 G_k(z) = \Gamma_j^2/[z - \varepsilon_k - \Sigma_k(z)],$$

which is a quadratic equation for $\Sigma_k(z)$ or, equivalently, for $G_k(z)$.

**Physical meaning:** Impurity system leads to a kind of damping, although it allows only for forward scattering and does not couple different momentum states by scattering processes. Impurities act like a viscous medium without genuine scattering.

$$\Sigma_k(z) = \frac{1}{2}(z - \varepsilon_k) - \sqrt{\frac{1}{4}(z - \varepsilon_k)^2 - \Gamma_j^2},$$

where the complex square root is defined as the principle branch, $\sqrt{z^2 - 1} = \sqrt{|z^2 - 1|} \exp(i \frac{1}{2} \arg(z^2 - 1))$, for $-\pi < \arg(z) < \pi$, so that for $|z| \gg 1$ one has $\sqrt{z^2 - 1} = z - 1/(2z) + O(|z|^{-3})$. Selfenergy and Green function are analytic in the complex plane with the exception of a branch cut along the interval $-2\Gamma_J \leq E - \varepsilon_k \leq 2\Gamma_J$ on the real axis. The spectral function $A_k(E) = (1/\pi) \text{Im} G_k(E - i0)$ is

$$A_k(E; \Gamma_J) = \frac{1}{\pi\Gamma_J} \sqrt{1 - \left(\frac{E - \varepsilon_k}{2\Gamma_J}\right)^2} \theta(2\Gamma_J - |E - \varepsilon_k|),$$

and has the shape of an half ellipse.
Impurity jelly

The corresponding real part is

\[
\text{Re } G_k(E - i0) = \begin{cases} 
\frac{(E - \varepsilon_k)}{2\Gamma_J}, & \text{if } |E - \varepsilon_k| \leq 2\Gamma_J \\
\frac{2}{[E - \varepsilon_k + \text{sign}(E - \varepsilon_k) \sqrt{(E - \varepsilon_k)^2 - 4\Gamma_J^2}]}, & \text{else.}
\end{cases}
\]

\[
\begin{align*}
\text{Figure:} & \quad \text{DOS of the free 2D EG, } D(E; 0) = D_0 \theta(E), \text{ and of the jelly model, } D(E; \Gamma_J), \text{ in units of } D_0 = m/(2\pi \hbar^2), \text{ together with the Green function } G_k^{-}(E) \text{ for } \\
& \quad \varepsilon_k = 20\Gamma_J. \\
& \quad \text{For } j > 2\Gamma_J: \quad D(E; \Gamma_J) = D(E; 0).
\end{align*}
\]
Short-range scatterers, \( B = 0 \)

Analytical considerations

opposite limit of impurities with very short-range potentials: \( v(r - R) = v_0 \delta(r - R) \).

Fourier coefficients \( v_\mathbf{q} = v_0 \) constant.

\[
\Sigma_k(z) = n_I \int \frac{d^2k'}{(2\pi)^2} v_{|k-k'|}^2 G_{k'}(z) \to n_I v_0^2 \int \frac{d^2k'}{(2\pi)^2} G_{k'}(z)
\]

\( \Sigma_k(E) \equiv \Sigma(E) \) independent of \( \mathbf{k} \), however \( k' \)-integral divergent in 2D (and 3D), cut-off required

possible procedure: subtract at finite range \( R \) an energy \( E_R \) from \( \text{Re}\Sigma \), so that for \( R \to 0 \) \( \text{Re}\Sigma - E_R \) remains finite

this motivates

\[
\Sigma(z) = n_I v_0^2 \Psi(z - \Sigma(z)), \quad \Psi(z) = \int \frac{d^2k}{(2\pi)^2} \left[ \frac{1}{z - \varepsilon_k} + \frac{1}{E_c + \varepsilon_k} \right],
\]

where \( E_c > 0 \) is a suitable cutoff energy. With \( D_0(E) = D_0 \theta(E) \), \( D_0 = m/(2\pi\hbar^2) \):

\[
\Psi(z) = D_0 \int_0^\infty d\varepsilon \left[ \frac{1}{z - \varepsilon} + \frac{1}{E_c + \varepsilon} \right] = D_0 \ln \left( \frac{-z}{E_c} \right).
\]

\( \Psi(z) \) is analytic in the complex plane with the exception of a branch cut along the positive real half-axis, where \( \text{Im}\Psi^-(E) = \text{Im}\Psi(E - i0) = \pi D_0(E) \).

Note that the impurity parameters \( v_0 \) and \( n_I \) enter only via the damping parameter \( \Gamma_0 = n_I v_0^2 D_0 \), of dimension energy.
Construction of analytic $\Sigma(z)$

define dimensionless quantities $\zeta = z/E_F$, $\sigma(\zeta) = \Sigma(z)/E_F$, $\varepsilon_c = E_c/E_F$, and $\gamma_0 = \Gamma_0/E_F$, for an arbitrary reference energy $E_F$:

$$\sigma(\zeta) = \gamma_0 \psi(\zeta - \sigma(\zeta)), \quad \psi(\zeta) = \ln(-\zeta/\varepsilon_c).$$

introduce a new complex variable $\tilde{\zeta}$ and the analytic functions

$$\tilde{\sigma}(\tilde{\zeta}) = \gamma_0 \psi(\tilde{\zeta}), \quad \tilde{\zeta}(\tilde{\zeta}) = \tilde{\zeta} + \tilde{\sigma}(\tilde{\zeta}).$$

If this has an inverse $\tilde{\zeta}(\zeta)$ that is analytic in the upper (lower) half of the $\zeta$-plane, then $\sigma(\zeta) = \tilde{\sigma}(\tilde{\zeta}(\zeta))$ is the required analytic selfenergy function.

We thus determine the region in the $\tilde{z} = \tilde{x} + i\tilde{y}$-plane that is mapped onto the upper half of the $\zeta = x + iy$-plane.

In $\tilde{y} > 0$:

$$\text{Im } \psi(\tilde{\zeta}) = \int_0^\infty d\varepsilon \frac{-\tilde{y}}{(\tilde{x} - \varepsilon)^2 + \tilde{y}^2} = -\left(\frac{\pi}{2} + \arctan \frac{\tilde{x}}{\tilde{y}}\right) < 0,$$

and $\text{Im } \zeta \equiv y = \tilde{y} + \gamma_0 \text{Im } \psi(\tilde{\zeta})$ is positive provided $\frac{\gamma_0}{\tilde{y}} \left(\frac{\pi}{2} + \arctan \frac{\tilde{x}}{\tilde{y}}\right) < 1$, which is certainly true for $\tilde{y}$ sufficiently large.

For $\tilde{x} < -\gamma_0$ and all $\tilde{y} > 0$ one finds $y > 0$, but $y = 0$ for $\tilde{y} = 0$, and we put $\tilde{y}(\tilde{x}) = 0$.

For $x > -\gamma_0$ we determine (numerically) the (largest) value $\tilde{y} = \tilde{y}(\tilde{x})$ that makes $y = 0$.

The function $\zeta(\tilde{\zeta})$ maps the region $\{-\infty < \tilde{x} < \infty, \tilde{y}(\tilde{x}) < \tilde{y} < \infty\}$ of the $\tilde{\zeta}$ plane onto the upper half of the $\zeta$ plane. We now show that it has an analytic inverse.
Parametric representation of $\Sigma^+(E)$

The derivative $d\zeta/d\tilde{\zeta} = 1 + \gamma_0 \psi'(\tilde{\zeta})$ does not vanish in the region $\tilde{y}(\tilde{x}) < \tilde{y}$, since $|\psi'(\tilde{\zeta})| < \int_0^\infty \frac{d\varepsilon}{|\zeta - \varepsilon|^2} = \frac{1}{\tilde{y}} \left( \frac{\pi}{2} + \arctan \frac{\tilde{x}}{\tilde{y}} \right) < \frac{1}{\gamma_0}$. Thus $\tilde{\zeta}(\zeta)$ has a local analytic inverse $\tilde{\zeta}(\zeta)$ everywhere in the specified region, and by analytic continuation we see that $\tilde{\zeta}(\zeta)$ is single valued and analytic in the upper half-plane. Along the real axis $\zeta = x + i0$ we obtain for the self-energy the parametric representation $(-\infty < \tilde{x} < \infty)$:

$$\sigma^+(x) = \sigma(x + i0) = \gamma_0 \psi(\tilde{x} + i\tilde{y}(\tilde{x})) = \frac{\gamma_0}{2} \ln \frac{\tilde{x}^2 + \tilde{y}(\tilde{x})^2}{\varepsilon_c^2} - i\tilde{y}(\tilde{x}),$$

$$x = \tilde{x} + \frac{\gamma_0}{2} \ln \frac{\tilde{x}^2 + \tilde{y}(\tilde{x})^2}{\varepsilon_c^2}.$$

Figure: Selfenergy for the 2D ES with $\delta$-impurities: $\text{Im } \Psi(E - \Sigma^-(E)) = \pi D(E; \Gamma_0) = \text{Im } \Sigma^-(E)/(n_I \nu_0^2)$ and the corresponding $\text{Re } \Psi(E - \Sigma^-(E)) = \text{Re } \Sigma^-(E)/(n_I \nu_0^2)$ versus energy $E$, for several values of the damping energy $\Gamma_0 = \gamma_0 E_F$. Cutoff energy $E_c = E_F$. 

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Green’s functions

Since \( \zeta(\zeta) \) is analytic in the upper half-plane with \( \text{Im} \zeta(\zeta) > 0 \), the Green function

\[
G_k(z) = \frac{1}{[\zeta(\zeta)E_F - \varepsilon_k]}
\]
is analytic in \( \text{Im}z > 0 \). Since \( |\Sigma(z)/z| \to 0 \) for \( |z| \to \infty \), the spectral function

\[
A_k(E) = -\text{Im}G_k(E + i0)/\pi
\]
has the correct normalization.

Figure: Selfenergy and spectral functions for the 2D ES with \( \delta \)-impurities: damping parameter \( \gamma_0 = \Gamma_0/E_F = 0.02 \), cutoff energy \( E_c = E_F \). Note asymmetry for \( |E| \leq \Gamma_0 \) and \( A_k(E) = 0 \) for \( E < -\Gamma_0[1 - 2 \ln(\Gamma_0/E_c)] \).
Simpler approximations

The lowest order Born approximation (LOBA)
\[ \Sigma^{-}(E) = n_1 \nu_0^2 \psi(E - \Sigma^{-}(E)) \approx n_1 \nu_0^2 \psi(E - i0) \]
- yields a good approximation to the SCBA results for high energies, \( E \gg \Gamma_0 \).
- yields quantitatively a poor results at \( E \sim \Gamma_0 \).
- breaks down exactly at the jump of the free electron DOS, \( E = 0 \), where \( \text{Re} \psi(E - i0) \) has a logarithmic singularity.

For \( E \sim \varepsilon_k \gg \Gamma_0 \), \( A_k(E) \) is well represented by the pole approximation for the selfenergy, \( \Sigma^+(E) \approx \Sigma^+(\varepsilon_k) \). Since the Green function \( G(E + i0) = i \int_0^\infty dE \exp(it(E + i0)) K(t) \) is the Laplace transform of the time evolution operator, this means that the free propagator \( K_k(t) = \exp(-it\varepsilon_k/\hbar) \) is replaced by \( K_k(t) = \exp(-it(\varepsilon_k + \Sigma^+(\varepsilon_k))/\hbar) \). Thus, the interaction with the impurity system leads to an energy shift \( \text{Re}\Sigma^+(\varepsilon_k) \) and to an exponential damping of the propagation described by a relaxation time \( \tau_k = \hbar/[\text{Im}\Sigma^+(\varepsilon_k)] \). The pole approximation for the selfenergy leads to a Lorentzian shape of the spectral function. The resulting DOS

\[
D_{\text{pole}}(E) = \frac{1}{\pi} \int_0^\infty d\varepsilon D_0(\varepsilon) \frac{-\text{Im}\Sigma^+(\varepsilon)}{(E - \varepsilon - \text{Re}\Sigma^+(\varepsilon))^2 + (\text{Im}\Sigma^+(\varepsilon))^2}
\]

has (in 2D and 3D ESs) a slowly decreasing, non-integrable tail at negative energies, which makes it useless for practical calculations.

In the 3DES at \( B = 0 \) the LOBA is completely equivalent to the SCBA.
Remark on need for self-consistency

We have seen that significant deviations between the simple Born approximation and the SCBA occur only near singularities (e.g. discontinuities) of the bare DOS.

Thus, the self-consistent BA will be necessary for physically meaningful results on the Landau quantized system, which has a bare DOS with $\delta$-function singularities.

On the other hand, the 3D ES for $B = 0$ has no discontinuity, and SCBA and simple BA are physically equivalent.
Example: 3D ES for $B = 0$

In analogy to the 2D case, we write in the SCBA

$$\Sigma(z) = n_1 v_0^2 \Psi(z - \Sigma(z)), \quad \Psi(z) = \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{z - \varepsilon_k} + \frac{1}{E_c + \varepsilon_k} \right],$$

in units of $E_F = \hbar^2 k_F^2 / 2m$, one obtains with $\zeta = z / E_F$, $\sigma(\zeta) = \Sigma(z) / E_F$, $\varepsilon_c = E_c / E_F$, and $\gamma_0 = n_1 v_0^2 k_F^3 / (4\pi E_F^2)$

$$\sigma(\zeta) = \gamma_0 \psi(\zeta - \sigma(\zeta)), \quad \psi(\zeta) = -i \sqrt{\zeta} - \sqrt{\varepsilon_c}.$$

Using the same approach as for the 2D case, one obtains [with $\sqrt{\Gamma_0} = \gamma_0 \sqrt{E_F} = n_1 v_0^2 (2m/\hbar)^{3/2} / 4\pi$]

$$\Sigma(E + i0) = \begin{cases} -i \sqrt{\Gamma_0 (E + \Gamma_0/4)}, & E > -\Gamma_0/4, \\ \Gamma_0 - \sqrt{\Gamma_0 (3\Gamma_0/4 - E)}, & E < -\Gamma_0/4. \end{cases}$$

Comparing with the lowest order Born approximation, we see that the SCBA leads to a rigid shift of the DOS along the energy axis, to the new band edge $-\Gamma_0/4$. 
We consider selfenergy and Green function of a homogeneous 2D ES in a strong perpendicular magnetic field

Landau representation: energy eigenstates $|n, k_y\rangle$, eigenfunctions

$$\langle r|n, k_y\rangle = \psi_{n,k_y}(x, y) = L_y^{-1/2} \exp(\imath y k_y) \varphi_n(x - X),$$

where $X = -l_c^2 k_y$, and energy eigenvalues $\varepsilon_n = \hbar \omega_c (n + 1/2)$ for $n = 0, 1, 2, \ldots$. We will assume impurity potentials with rotation symmetry in the plane of the 2D ES: $\nu_q = \nu_q$ depend only on the modulus $q = |q|$.

Then one can proof that the Green function, and as a consequence the selfenergy, are diagonal in the Landau representation,

$$\langle n', k'_y|\hat{G}(z)|n, k_y\rangle = \delta_{k'_y, k_y} \delta_{n', n} G_n(z),$$

and the matrix elements are independent of the center coordinate $X$ (i.e. on $k_y$).

[see Gerhardts, Z. Physik 22, 327 (1975), and earlier references therein]

We will not give the full prove, but show that a solution with this properties exists.
Relevant matrix elements in Landau representation

The potential matrix elements in Landau representation are

\[ \langle m, k_y | V_R(\hat{r}) | n, k'_y \rangle = \sum_{q} \frac{V_q}{J} e^{-i q \cdot \hat{R}} \langle m, k_y | e^{i q \cdot \hat{r}} | n, k'_y \rangle, \quad (1) \]

those of the plane-wave factors [see D. Pfannkuche et al., Phys. Rev. B 46, 12606 (1992)]

\[ \langle n', k'_y | e^{i q \cdot \hat{r}} | n, k_y \rangle = \delta_{k'_y, k_y + q_y} e^{-\frac{i}{2} l_c^2 q_x (k'_y + k_y)} \mathcal{L}_{n'n}(q), \quad (2) \]

with

\[ \mathcal{L}_{n'n}(q) = i^{n'-n} \left( \frac{q_x + i q_y}{|q|} \right)^{n-n'} \mathcal{J}_{n'n}(Q), \quad Q = l_c^2 q^2 / 2. \quad (3) \]

\[ \mathcal{J}_{n'n}(Q) = \sqrt{\frac{\tilde{n}!}{n'! n!}} Q^{1/2 |n' - n|} e^{-1/2 Q} L_{\tilde{n}}^{(|n' - n|)}(Q), \quad L_{n}^{(\alpha)}(x) = \sum_{m=0}^{n} \binom{n + \alpha}{n - m} \frac{(-x)^m}{m!}, \quad (4) \]

where \( \tilde{n} = \min(n', n) \) is the minimum of \( n' \) and \( n \), and \( L_{n}^{(\alpha)}(x) \) is a generalized Laguerre polynomial. Note the symmetry relations

\[ \mathcal{J}_{n'n}(Q) = \mathcal{J}_{nn'}(Q), \quad \mathcal{L}_{n'n}(-q) = [\mathcal{L}_{nn'}(q)]^*. \quad (5) \]
Derivation of selfenergy equation in Landau representation

Inserting the Fourier expansion of the impurity potential into the SCBA selfenergy equation \( \hat{\Sigma}(z) = n_I \int d^2 R \hat{\nu}_R \hat{G}(z) \hat{\nu}_R \), we can immediately perform the \( R \)-integration, \( \int d^2 R \exp(-i[q + q'] \cdot R) = \mathcal{F} \delta_{q',-q} \), to obtain

\[
\left\langle m', k_y' \left| \hat{\Sigma}(z) \right| n', k_y'' \right\rangle = n_I \sum_q \frac{|v_q|^2}{\mathcal{F}} \sum_{m,n,k_y} \left\langle m', k_y' \left| e^{iq \cdot \hat{f}} \right| m, k_y \right\rangle \\
\times \ G_{m,n}(z; X) \left\langle n, k_y \left| e^{-iq \cdot \hat{f}} \right| n', k_y'' \right\rangle ,
\]

where we have exploited the translational invariance in \( y \) direction and used the notation \( \left\langle m, k_y \left| \hat{G}(z) \right| n, k_y' \right\rangle = \delta_{k_y,k_y'} G_{m,n}(z; X) \), allowing for the moment for a possible dependence on the center coordinate \( X = -l_c^2 k_y \); Similar \( \left\langle m, k_y \left| \hat{\Sigma}(z) \right| n, k_y' \right\rangle = \delta_{k_y,k_y'} \Sigma_{m,n}(z; X) \). The remaining Kronecker \( \delta_{k_y', k_y + q_y} \) can be used to eliminate either the sum over \( q_y \) or that over \( k_y \). Now we can take the limit \( \mathcal{F} \to \infty \) and write

\[
\Sigma_{m',n'}(z; X') = \sum_{m,n} \int_{-\infty}^{\infty} dX \ \mathcal{W}_{m,n}^{m',n'}(X' - X) \ G_{m,n}(z; X) ,
\]

\[
\mathcal{W}_{m,n}^{m',n'}(X' - X) = \int \frac{d^2 q}{2\pi} \ \frac{n_I |v_q|^2}{2\pi l_c^2} \ L_{m'm}(q) L_{nn'}(-q) \delta \left( q_y + \frac{X' - X}{l_c^2} \right) .
\]

If \( G_{m,n}(z; X) \) is independent of \( X \), as is known for the homogeneous Landau system the \( X \)-integration can be performed trivially and the result becomes much simpler:
Selfenergy equation in Landau representation

If \( G_{m,n}(z;X) \) is independent of \( X \), \( \Sigma_{m',n'}(z;X') \) turns out to be independent of \( X' \). We now can perform the two-dimensional \( q \)-integral taking advantage of the rotational symmetry of the impurity potential, \( v_q = v_q \). With polar coordinates, \( q_x + iq_y = q \exp(i\varphi) \), we see that the \( \varphi \)-integral can be performed and yields a conservation rule for the Landau quantum numbers. The result can be written in the form

\[
\Sigma_{m',n'}(z) = \sum_{m,n} W_{m,n}^{m',n'} G_{m,n}(z) \tag{9}
\]

\[
W_{m,n}^{m',n'} = \delta_{m'-n',m-n} \int_0^{\infty} dQ \left[ \frac{n_l|v_q|^2}{2\pi l_c^2} \right]_{q=\frac{\sqrt{2Q}}{l_c}} J_{m'm}(Q) J_{nn'}(Q) \tag{10}
\]

With the ansatz that the Green operator \( \hat{G}(z) \) is diagonal in the Landau representation, \( G_{m,n}(z) = \delta_{m,n} G_m(z) \), we see that the selfenergy is also diagonal, \( \Sigma_{m,n}(z) = \delta_{m,n} \Sigma_m(z) \) where

\[
\Sigma_m(z) = \sum_n \Gamma_{m,n}^2 G_n(z) , \quad \Gamma_{m,n}^2 = \int_0^{\infty} dQ \frac{n_l|v_{\sqrt{2Q}/l_c}|^2}{2\pi l_c^2} [J_{mn}(Q)]^2 . \tag{11}
\]

We thus have shown that the SCBA has a solution which is diagonal in the Landau representation and independent of the center coordinate, and we know from the literature that this is the only solution.
**Jelly model**

For the jelly model we evaluate

\[
\frac{n_i}{\mathcal{F}} \sum_{\mathbf{q}} |v_{\mathbf{q}}|^2 M(\mathbf{q}) \Rightarrow \frac{n_i}{(2\pi)^2} \int d^2 q |v_{\mathbf{q}}|^2 M(\mathbf{q}) = \int d^2 q \Gamma_j^2 \delta(\mathbf{q}) M(0)
\]

(where \( \Gamma_j \) is independent of \( B \)), so that the matrix elements reduce to the orthonormality relations of the Landau states. Taking the diagonality of the Green's function for granted, the selfenergy equation reduces to

\[
\Sigma_n(z) = \Gamma_j^2 G_n(z) = \Gamma_j^2 / [z - \varepsilon_n - \Sigma_n(z)],
\]

i.e., apart from the occurrence of the Landau energy, the same quadratic equation as in the limit \( B = 0 \).

- Selfenergy, spectral function and real part of the Green function are as for \( B = 0 \), but with \( k \) replaced by the Landau quantum number \( n \).
- For \( 4\Gamma_j < \hbar \omega_c \), the spectral functions of different Landau levels do not overlap, and the DOS consists of equidistant peaks of the the form of semi-ellipses.
- This lack of overlap of the spectral functions belonging to adjacent LLs, will lead to the pathological result of vanishing longitudinal conductivity.
- Note that for this model of infinite-range impurity potentials the selfenergy \( \Sigma_n^{-}(E) \) manifestly depends on the Landau quantum number \( n \), since its imaginary part \( \text{Im} \Sigma_n^{-}(E) = \pi \Gamma_j^2 A_n(E) \) is nonzero only for \( |E - \varepsilon_n| < 2\Gamma_j \).
Selfenergy due to short-range scatterers

With \( v_q = v_0 \), the \( Q \)-integral in

\[
\Gamma^2_{m,n} = \int_0^\infty dQ \frac{n_l |v \sqrt{2Q/l_c}|^2}{2\pi l_c^2} \left[ J_{mn}(Q) \right]^2 \equiv \Gamma^2_B = n_l v_0^2/(2\pi l_c^2)
\]

reduces to the normalization of the generalized Laguerre polynomials,

\[
\int_0^\infty dQ \left[ J_{mn}(Q) \right]^2 = 1,
\]

and the \( \Gamma^2_{m,n} \) become independent of the Landau quantum numbers. To handle the resulting singularity in real part of the selfenergy, we use the same kind of cutoff as as for \( B = 0 \), replacing the free electron DOS by the Landau DOS, and define

\[
\Psi(z) = \frac{1}{2\pi l_c^2} \sum_{n=0}^\infty \left[ \frac{1}{z - \varepsilon_n} + \frac{1}{E_c + \varepsilon_n} \right] = D_0 \left[ \psi \left( \frac{1}{2} - \frac{z}{\hbar \omega_c} \right) - \psi \left( \frac{1}{2} + \frac{E_c}{\hbar \omega_c} \right) \right],
\]

where \( \varepsilon_n = \hbar \omega_c (n + 1/2) \) is a Landau energy, \( \psi(z) = d(\ln \Gamma(z))/dz \) is the digamma function, and \( D_0 = m/(2\pi \hbar^2) \). The selfenergy equation for \( \delta \)-scatteres in the SCBA then reads

\[
\Sigma(z) = n_l v_0^2 \Psi(z - \Sigma(z)).
\]

The solution is obviously independent of the Landau quantum number \( n \).
Solution satisfying analytic requirements

Since we need only $\Sigma(E - i0^+) \text{ next to the real axis, we may try to calculate this for each $E$ by direct iteration and hope, that the obtained solution yields normalized spectral functions.} 

To be sure about this, we may proceed as for $B = 0$: 

Measuring energies in units of the cyclotron energy, $z/\hbar \omega_c = \zeta$, $\Sigma(z)/\hbar \omega_c = \sigma(\zeta)$, $E_c/\hbar \omega_c = \varepsilon_c$, the selfenergy equation reads 

$$\sigma(\zeta) = \gamma_B^2 \left\{ \psi(1/2 - [\zeta - \sigma(\zeta)]) - \psi(1/2 + \varepsilon_c) \right\},$$

where the damping parameter is now 

$$\gamma_B^2 = n_I v_0^2 D_0 / \hbar \omega_c = (\Gamma_B / \hbar \omega_c)^2, \quad \Gamma_B^2 = \Gamma_0 \hbar \omega_c, \quad \Gamma_0 = n_I v_0^2 D_0.$$

We consider the $\sigma(\zeta) = \tilde{\sigma}(\tilde{\zeta})$ as function of the complex variable $\tilde{\zeta} = \zeta - \sigma(\zeta)$ in the upper halfplane $\text{Im} \tilde{\zeta} > 0$. The boundary of the region in which the function $\zeta(\tilde{\zeta}) = \tilde{\zeta} + \tilde{\sigma}(\tilde{\zeta})$ has a unique, analytic inverse corresponds to the real $\zeta$-axis and is given by $\tilde{\zeta} = \tilde{x} + i\tilde{y}(\tilde{x})$, where $\tilde{y}(\tilde{x})$ is the solution $\tilde{y}$ of 

$$\gamma_B^2 \sum_{n=0}^{\infty} \frac{1}{(n + 1/2 - \tilde{x})^2 + \tilde{y}^2} = 1,$$

if such a solution exists for the considered $\tilde{x}$-value, and $\tilde{y}(\tilde{x}) = 0$ else. This yields the following parametric representation of the selfenergy:
Parametric representation of selfenergy

\[ x = \tilde{x} + \text{Re}\tilde{\sigma}(\tilde{x} + i\tilde{y}(\tilde{x})), \quad \sigma(x + i0^+) = \tilde{\sigma}(\tilde{x} + i\tilde{y}(\tilde{x})), \]

where \( \text{Im}\tilde{\sigma}(\tilde{x} + i\tilde{y}(\tilde{x})) = -\tilde{y}(\tilde{x}). \)

Figure: Real (upper) and imaginary part (lower panel) of the selfenergy, scaled as
\[ \Sigma^{-}(E)/(\pi\Omega\gamma_{B}^2) = \Psi(E - \Sigma^{-}(E))/(\pi D_{0}) \text{ vs } E/\Omega \]
for \( \Omega = \hbar\omega_{c} \) and the four values of the damping parameter \( \gamma_{B}^2 \) indicated in the legend. For \( \gamma_{B}^2 = 0.01 \) and 0.05, the DOS consists of separated peaks centered near the Landau energies \( E = \varepsilon_{n} = \hbar\omega_{c}(n + 1/2). \)

In the intervals of finite DOS, the real part of the selfenergy increases nearly linearly with energy. For larger damping, \( \gamma_{B}^2 \gtrsim 0.1 \), the DOS peaks overlap, so that an oscillating DOS without zeroes results. For still larger damping these oscillations die out and for \( \gamma_{B}^2 \gtrsim 0.5 \) the DOS and the real part of \( \Psi(E - \Sigma^{-}(E)) \) assume the form known from \( B = 0 \), without any visible magnetic quantum oscillations.
Spectral functions

Figure: Spectral functions (solid lines) and corresponding real parts $\Re \, G_n^-(E)/\pi$ (dash-dotted lines) of the lowest Landau levels, calculated for $\gamma_B^2 = 0.05$, $\Omega = \hbar \omega_c$. Since the selfenergy $\Sigma^-(E)$ is independent of the Landau quantum number $n$, the Green function $G_n^-(E)$ has finite tails at the energies of other Landau levels, $E \approx \varepsilon_{n'}$ for $n' \neq n$.

For small damping, the side peaks of the spectral functions are small,

$$\pi A_n(\varepsilon_{n+1}) = \Im \Sigma^-(\varepsilon_{n+1})/|\varepsilon_{n+1} - \varepsilon_n - \Sigma^-(\varepsilon_{n+1})|^2 \approx \Im \Sigma^-(\varepsilon_n)/(\hbar \omega_c)^2$$

while

$$\pi A_n(\varepsilon_n) \approx 1/\Im \Sigma^-(\varepsilon_n),$$

so that $A_n(\varepsilon_{n+1})/A_n(\varepsilon_n) \sim (\Im \Sigma^-(\varepsilon_n)/\hbar \omega_c)^2$. Nevertheless, the overlap of the spectral functions for adjacent Landau levels, produced by these side peaks, will turn out to be essential for the calculation of conductivities.

For weak damping (“weak collision broadening”), the coupling between LLs may be treated perturbatively.
Spectral function perturbatively

For weak collision broadening, the spectral functions become narrow and the side peaks at the energies of adjacent Landau levels become very small. If we neglect the coupling between LLs completely, we obtain for the LL $n$ the quadratic equation

$$
\sigma(\zeta) = \frac{\gamma_B^2}{\zeta - (n + 1/2) - \sigma(\zeta)} + c_n,
$$

where

$$
c_n = \frac{\gamma_B^2}{\epsilon_c + (n + 1/2)}.
$$

The relevant solution along the real $\zeta$-axis is

$$
\sigma_n(x - i0^+) = \begin{cases} 
z_n^+ + i\sqrt{4\gamma_B^2 - (z_n^-)^2}, & \text{if } |z_n^-| < 2\gamma_B, \\
z_n^+ - z_n^- \sqrt{1 - 4\gamma_B^2 / (z_n^-)^2}, & \text{if } |z_n^-| \geq 2\gamma_B,
\end{cases}
$$

where $z_n^\pm = x - (n + 1/2) \pm c_n$. The simplest approximation for the spectral function is

$$
A_n^{(0)}(E) = \text{Im}\sigma_n(E/\hbar\omega_c - i0^+)/\left(\pi\gamma_B^2\hbar\omega_c\right).
$$

For $n = 1$ this approximation is indicated in following figure by the large peak plotted as a dash-dotted line (without the side peaks near the LLs with $n \neq 1$). Since $\sigma_n(\zeta)$ is analytic in the upper (lower) halfplane, $A_n^{(0)}(E)$ has the correct normalization.

To obtain the side peaks we define the $n$-independent selfenergy $\sigma(\zeta) = \sum_{n=0}^\infty \sigma_n(\zeta)$, which is well defined (convergent) and analytic due to the presence of the $c_n$ terms. Therefore a reasonable, correctly normalized approximation for the spectral function would be

$$
A_n^{(\sigma)}(E) = \frac{1}{\pi} \text{Im} \frac{1}{E - \hbar\omega_c[n + 1/2 + \sigma(E/\hbar\omega_c - i0^+)]}.
$$
Approximations for the spectral function

**Figure:** Approximations of the spectral function \( A_n(E) \) for \( n = 1 \): exact numerical result (solid line), analytic approximation \( A_n^{(\sigma)}(E) \) (dashed line), and perturbative approximation (dash-dotted line; without side peaks: solution \( A_n^{(0)}(E) \) of quadratic approximation); \( \gamma_B^2 = 0.05 \), \( \Omega = \hbar \omega_c \).

A simpler approximation is obtained if one considers the terms \( \sigma_{n'}(\zeta) \) with \( n' \neq n \) to lowest order: \( A_n^{(1)}(E) = A_n^{(0)}(E) + \sum_{n'(\neq n)} \Gamma_B^2 A_{n'}^{(0)}(E)/(E - \varepsilon_n)^2 \). The last term yields side peaks similar to the exact result (dash-dotted lines). The energy integral over \( A_n^{(1)}(E) \) deviates from unity by terms of the order \( (\Gamma_B/\hbar \omega_c)^2 \). If one replaces in the denominator of the last term \( E \) by \( \varepsilon_{n'} \), the side peaks near the other LLs become symmetric.
Finite-range scatterers

- If the impurity potential is of finite range, the coupling of LLs is given by the $\Gamma_{m,n}$ and must be calculated numerically.

- If the coupling is weak (weak collision broadening), one may neglect it in a first step and calculate the selfenergy from the quadratic equation
  \[ \Sigma_n^{(0)}(z) = \Gamma_{n,n}^2 [z - \varepsilon_n - \Sigma_n^{(0)}(z)]^{-1} \equiv \Gamma_{n,n}^2 G_n^{(0)}(E). \]
  In a next step, one may proceed with $\Sigma_n^{(1)}(z) = \sum_m \Gamma_{n,m}^2 G_m^{(0)}(E)$, etc.

- An advantage of finite-range scatterers, besides of being more realistic than $\delta$-potential scatterers, is that the selfenergy is finite and no cutoff or renormalization procedure is needed.
Exact formulas for the conductivity tensor

Kubo’s formulas yield for the spatially resolved, static linear response to a constant electric field the conductivity tensor

\[
\sigma_{\mu\nu}(q, q' = 0; \omega = 0) = \lim_{\omega \to 0} \frac{1}{\omega} \left[ \chi_{\mu\nu}(q, 0; \omega) - \chi_{\mu\nu}(q, 0; 0) \right].
\] (12)

Inserting the impurity-averaged vertex functions, we obtain in the limit \( q \to 0 \) for \( \sigma_{\mu\nu} \equiv \sigma_{\mu\nu}(q = 0, 0; 0) \), and with \( \hat{j}^0_{\nu} = -e\hat{v}_{\nu}/\mathcal{F} \),

\[
\sigma_{\mu\nu} = \frac{\hbar\mathcal{F}}{2\pi} \int_{-\infty}^{\infty} dE \, f(E) \, \text{Sp} \, \hat{j}^0_{\mu} \left[ \frac{\partial}{\partial E} \hat{F}^{+-}(\hat{j}^0_{\nu}; E, E') - \frac{\partial}{\partial E} \hat{F}^{+-}(\hat{j}^0_{\nu}; E, E') \right.
\]

\[
\left. - \frac{\partial}{\partial E'} \hat{F}^{--}(\hat{j}^0_{\nu}; E, E') + \frac{\partial}{\partial E'} \hat{F}^{+-}(\hat{j}^0_{\nu}; E, E') \right]_{E' = E}.
\]

This can be written as

\[
\sigma_{\mu\nu} = \int_{-\infty}^{\infty} dE \left( -\frac{df}{dE} \right) \sigma^{(I)}_{\mu\nu}(E) + \sigma^{(II)}_{\mu\nu}, \quad \text{with}
\]

\[
\sigma^{(I)}_{\mu\nu}(E) = \frac{e^2 \hbar}{2\pi \mathcal{F}} \, \text{Sp} \, \hat{v}_\mu \left\{ \hat{F}^{+-}(\hat{v}_\nu) - \frac{1}{2} \left[ \hat{F}^{--}(\hat{v}_\nu) + \hat{F}^{+-}(\hat{v}_\nu) \right] \right\}
\]

and

\[
\sigma^{(II)}_{\mu\nu} = \frac{e^2 \hbar}{4\pi \mathcal{F}} \int_{-\infty}^{\infty} dE \, f(E) \, \text{Sp} \, \hat{v}_\mu \left[ \hat{D}^-(\hat{v}_\nu) - \hat{D}^+(\hat{v}_\nu) \right],
\]

where

\[
\hat{F}^{\sigma\sigma'}(\hat{v}_\nu) = \hat{F}^{\sigma\sigma'}(\hat{v}_\nu; E, E), \quad \hat{D}^\sigma(\hat{v}_\nu) = \left[ \left( \frac{\partial}{\partial E} - \frac{\partial}{\partial E'} \right) \hat{F}^{\sigma\sigma}(\hat{v}_\nu; E, E') \right]_{E' = E}.
\]
Reduction to the SCBA

The \( \hat{F}^{\sigma\sigma'}(\hat{\nu}_\nu; E, E') = \hat{F}(\hat{\nu}_\nu; E + i\sigma 0^+, E' + i\sigma' 0^+) \) must be calculated from the Bethe-Salpeter equation, which reads in the present notation and in the SCBA

\[
\hat{F}^{\sigma\sigma'}(\hat{\nu}_\nu) = \hat{G}^\sigma \left\{ \hat{\nu}_\nu + \mathcal{I} \left[ \hat{F}^{\sigma\sigma'}(\hat{\nu}_\nu) \right] \right\} \hat{G}^{\sigma'}, \quad \text{with} \quad \mathcal{I}[\hat{\nu}] = n_l \int d^2 R \hat{\nu}_R \hat{\nu}_R.
\]

With the definition

\[
\hat{F}^{\sigma\sigma}(\hat{\nu}_\nu) = \hat{G}^\sigma \hat{\Lambda}^{\sigma\sigma}_\nu \hat{G}^{\sigma},
\]

we may write

\[
\hat{D}(\hat{\nu}_\nu) = \frac{d\hat{G}}{dE} \hat{\Lambda}_\nu \hat{G} - \hat{G} \hat{\Lambda}_\nu \frac{d\hat{G}}{dE} + \hat{G} \left\{ \mathcal{I} \left[ \hat{D}(\hat{\nu}_\nu) \right] \right\} \hat{G},
\]

where the equal superscripts (\( \sigma \) or \( \sigma' \)) have been suppressed.

According to the cyclic invariance of the trace, one has

\[
\text{Sp} \hat{\nu}_\mu \hat{F}(\hat{\nu}_\mu; z, z') = \text{Sp} \hat{\nu}_\mu \hat{F}(\hat{\nu}_\mu; z', z),
\]

and the \( \hat{D}^{\sigma}(\hat{\nu}_\nu) \) do not contribute to the diagonal part of the conductivity, \( \sigma^{(\|)}_{\mu\mu} = 0 \).

It is not necessary to solve integral equations for the \( \hat{F}^{\sigma\sigma}(\hat{\nu}_\nu) \) contributing to \( \sigma^{(\|)}(E) \), since they can be obtained from the Ward identities

\[
\hat{F}^{\sigma\sigma}(\hat{\nu}_\nu) = \frac{i}{\hbar} \left[ \hat{G}^\sigma , \hat{\nu}_\nu \right].
\]
Conductivity for $B = 0$

To understand the role of vertex corrections, we first consider the case $B = 0$.

- We work in the momentum representation, in which the free-electron Hamiltonian and the velocity operator are diagonal, $\langle k' | H_0 | k \rangle = \epsilon(k) \delta_{k', k}$, $\langle k' | \hat{v} | k \rangle = (\hbar k / m) \delta_{k', k}$.

- It is easy to show that the vertex operator $\hat{F}(\hat{v}; z, z')$ is also diagonal,
  $$\langle k' | \hat{F}(\hat{v}; z, z') | k \rangle = F_{k}(\hat{v}; z, z') \delta_{k', k}.$$  

- To prove this, note that the velocity operator $\hat{v} = \hat{p} / m$ commutes with the translation operator $\hat{T}_a = \exp(i a \cdot p / \hbar)$ and then proceed as we did when we proved that the Green operator $\hat{G}(z)$ is diagonal.

With $\langle k' | \hat{L}[\hat{\sigma}] | k \rangle = \sum_{p', p} W_{p', p}^{k', k} \langle p' | \hat{\sigma} | p \rangle$, $W_{p', p}^{k', k} = \delta_{k', k} - k, p' - p \frac{n_l}{F} \left| v_{k'} - p \right|^2$,

and the notation $F_{k}(\hat{v}_{\nu}; E + \sigma i 0^+, E + \sigma' i 0^+) = F_{\nu}^{\sigma, \sigma'}(k)$, the Bethe-Salpeter equation becomes

$$F_{\nu}^{\sigma, \sigma'}(k) = G_{k}^{\sigma} \left\{ \frac{\hbar k_{\nu}}{m} + n_l \int \frac{d^2 p}{(2\pi)^2} \left| v_{k} - p \right|^2 F_{\nu}^{\sigma, \sigma'}(p) \right\} G_{k}^{\sigma'}$$
Use of Ward identities

Since selfenergy $\Sigma_k(z) = \Sigma_{-k}(z)$ and Green function $G_k(z) = G_{-k}(z)$ are even functions of $k$ and depend only on $\varepsilon_k = (\hbar k)^2 / 2m$, the vertex functions $F_{\nu}^{\sigma,\sigma'}(k) = -F_{\nu}^{\sigma,\sigma'}(-k)$ are odd functions of $k$, just as the velocity matrix elements $v_\nu = \hbar k_\nu / m$.

Since in the momentum representation the position operator reads $\hat{\mathbf{r}} = i\hbar \partial / \partial \hat{\mathbf{p}}$, the Ward identity $\hat{F}^{\sigma\sigma}(\hat{v}_\mu; E) = (i/\hbar)[\hat{G}^{\sigma}(E), \hat{v}_\mu]$, reads in the momentum representation $\hat{F}^{\sigma\sigma}(\hat{v}_\mu; E) = \partial \hat{G}^{\sigma}(E) / \partial \hat{p}_\mu$. Taking matrix elements, this yields

$$F_{\nu}^{\sigma,\sigma'}(k) = \frac{1}{\hbar} \frac{\partial G_k^\sigma}{\partial k_\nu} = \frac{\hbar k_\nu}{m} \frac{\partial G_k^\sigma}{\partial \varepsilon_k} = \left[ \frac{\hbar k_\nu}{m} + \frac{1}{\hbar} \frac{\partial \Sigma_k^\sigma}{\partial k_\nu} \right] (G_k^\sigma)^2.
$$

Thus, we do not need to solve the B-S-equation

$$F_{\nu}^{\sigma,\sigma'}(k) = G_k^\sigma \left\{ \frac{\hbar k_\nu}{m} + n_l \int \frac{d^2p}{(2\pi)^2} |v_{k-p}|^2 F_{\nu}^{\sigma,\sigma'}(p) \right\} G_k^{\sigma'}$$

for $\sigma' = \sigma$. Inserting this Ward identity into the B-S-equation, we obtain

$$\frac{\partial \Sigma_k^\sigma}{\partial k_\nu} = n_l \int \frac{d^2p}{(2\pi)^2} |v_{k-p}|^2 \frac{\partial G_p^\sigma}{\partial p_\nu}$$
Jelly model

We have already mentioned that the jelly model, defined by

\[ n_{\|} v_{q}^{2} / (2\pi)^{2} = \Gamma_{0}^{2} \delta(q), \]

leading to

\[ \Sigma_{k}(z) = \Gamma_{0}^{2} G_{k}(z) = \Gamma_{0}^{2} / [z - \varepsilon_{k} - \Sigma_{k}(z)], \]

is not useful for transport calculations. To see that, we note that the solution of this quadratic equation for \( z = E \pm i0^{+} \) with \((E - \varepsilon_{k})^{2} < 4\Gamma_{0}\) is

\[ \Sigma_{k}^{\pm} = \frac{1}{2} (E - \varepsilon_{k}) \mp i \sqrt{\Gamma_{0}^{2} - \frac{1}{4} (E - \varepsilon_{k})^{2}}, \]

so that

\[ G_{k}^{+} G_{k}^{-} = \Sigma_{k}^{+} \Sigma_{k}^{-} / \Gamma_{0}^{4} = 1 / \Gamma_{0}^{2}. \]

Now consider the B-S-equation for \( \sigma = -\sigma' = + \) and multiply by \( [G_{k}^{+} G_{k}^{-}]^{-1} \):

\[ F_{\nu}^{+,-}(k) / [G_{k}^{+} G_{k}^{-}] = \frac{\hbar k_{\nu}}{m} + \Gamma_{0}^{2} F_{\nu}^{+,-}(k). \]

We conclude that this has no finite solution \( F_{\nu}^{+,-}(k) \). As a consequence we find that the jelly model does not lead to finite conductivity, and we interpret this result as being due to the absence of genuine scattering in this model.
Resistance due to short-range impurities

Next we consider the opposite case of short-range impurity potentials with constant Fourier coefficients, so that the integral in the Bethe-Salpeter equation vanishes for parity reasons. As a consequence, the vertex functions

\[ F^{\sigma,\sigma'}_{\nu} (k) = (\hbar k_{\nu} / m) \, G^\sigma_k(E) \, G^{\sigma'}_k(E) \]

can be expressed in terms of the selfenergy, and no genuine transport equation has to be solved. Thus, the conductivity formula

\[ \sigma^{(l)}_{\mu \nu}(E) = \frac{e^2 \hbar}{2 \pi \mathcal{F}} \text{Sp} \, \hat{\nu}_\mu \left\{ \hat{F}^{+-}(\hat{\nu}_\nu) - \frac{1}{2} \left[ \hat{F}^{--}(\hat{\nu}_\nu) + \hat{F}^{++}(\hat{\nu}_\nu) \right] \right\} \]

can be written as

\[
\sigma_{\mu \nu}(E) = \frac{e^2 \hbar}{2 \pi} g_s \int \frac{d^2 k}{(2\pi)^2} \frac{\hbar k_{\mu}}{m} \frac{\hbar k_{\nu}}{m} \left\{ G^+_k G^-_k - \frac{1}{2} \left[ (G^+_k)^2 + (G^-_k)^2 \right] \right\} \\
= \frac{e^2 g_s}{\hbar} \int_0^\infty d\varepsilon_k \, \varepsilon_k \, \frac{1}{2} [A_k(E)]^2 , \quad \text{since} \quad G^-_k - G^+_k = 2i\pi A_k(E). \]

Note that the \( \hat{F}^{--} \) and \( \hat{F}^{++} \) terms ensure the convergence of the integrals and may not be neglected.
Drude-type formula

To evaluate the integral, we note that $\Sigma^{\pm}(E) = \Delta(E) \mp (i/2)\Gamma(E)$ is independent of $k$ and write

$$\varepsilon_k[A_k(E)]^2 = \frac{\varepsilon_k}{(2\pi i)^2}[(G_k^-)^2 - 2G_k^-G_k^++(G_k^+)^2]$$

$$= \frac{1}{(2\pi i)^2} \left\{ \frac{E - \Sigma^- - (E - \varepsilon_k - \Sigma^-)}{(E - \varepsilon_k - \Sigma^-)^2} + \frac{E - \Sigma^+ - (E - \varepsilon_k - \Sigma^+)}{(E - \varepsilon_k - \Sigma^+)^2} - \frac{2\varepsilon_k}{i\Gamma}[G_k^- - G_k^+] \right\}$$

$$= \frac{1}{(2\pi i)^2} \left\{ (E - \Sigma^-)(G_k^-)^2 + (E - \Sigma^+)(G_k^+)^2 - \frac{2}{i\Gamma}G_k^-[\varepsilon_k + i\frac{\Gamma}{2}] + \frac{2}{i\Gamma}G_k^+[\varepsilon_k - i\frac{\Gamma}{2}] \right\}$$

$$= \frac{1}{(2\pi i)^2} \left\{ (E - \Sigma^-) \frac{\partial G_k^-}{\partial \varepsilon_k} + (E - \Sigma^+) \frac{\partial G_k^+}{\partial \varepsilon_k} - \frac{2}{i\Gamma}(E - \Delta)(G_k^- - G_k^+) \right\} .$$

$$\int_0^\infty d\varepsilon_k \varepsilon_k[A_k(E)]^2 = \frac{1}{(2\pi i)^2} \left[ \frac{E - \Sigma^-}{E - \varepsilon_k - \Sigma^-} + \frac{E - \Sigma^+}{E - \varepsilon_k - \Sigma^+} \right]_{\varepsilon_k=0}^{\varepsilon_k=\infty} + \frac{E - \Delta}{\pi \Gamma} \int_0^\infty d\varepsilon_k A_k(E)$$

Finally we obtain

$$\sigma_{\mu\mu}(E) = \frac{e^2}{m} \frac{\hbar}{D_0} \left\{ \frac{1}{2\pi} + \frac{E - \Delta(E)}{\Gamma(E)} \right\} \int_0^\infty d\varepsilon_k A_k(E) ,$$

where $D_0 = g_s m/(2\pi \hbar^2)$. With $\tau(E) = \hbar/\Gamma(E)$ this can be rewritten as

$$\sigma_{\mu\mu}(E) = \frac{e^2}{m} \tau(E) \tilde{n}(E) , \quad \text{with} \quad \tilde{n}(E) = [E - \Delta(E)] D(E) + \frac{1}{2\pi} D_0 \Gamma(E) ,$$

where the density of states is given by

$$D(E) = D_0 \int_0^\infty d\varepsilon_k A_k(E) = \frac{D_0}{\pi} \left\{ \frac{\pi}{2} + \arctan \left( \frac{E - \Delta(E)}{\Gamma(E)/2} \right) \right\} .$$
Drude formula for conductivity at $B = 0$

It is easy to show that, for $B = 0$ and within the SCBA,

\[ \tilde{n}(E) = \int_{-\infty}^{E} dE' \ D(E') \]

is the integrated density of states [which is definitely not true for $B \neq 0$]. To prove this, it is sufficient to show that

\[ \frac{d\tilde{n}}{dE} - D(E) = -\frac{d\Delta}{dE} \ D(E) + [E - \Delta(E)] \frac{dD}{dE} + \frac{D_0}{2\pi} \frac{d\Gamma}{dE} \]

vanishes. To demonstrate this, we first express the density of states by the SCBA relation $D(E) = \Gamma(E)D_0/(2\pi\Gamma_0)$, and then use the basic SCBA relation again to calculate the derivative,

\[ \frac{d\Sigma^-}{dE} = -\Gamma_0 \int_{0}^{\infty} d\varepsilon \frac{1 - d\Sigma^-/dE}{[E - \varepsilon - \Sigma^-(E)]^2} = \frac{\Gamma_0}{E - \Sigma^-(E) + \Gamma_0} \]

Using this to calculate $d\Delta/dE = \text{Re}[d\Sigma^-/dE]$ and $d\Gamma/dE = 2\text{Im}[d\Sigma^-/dE]$ proves $d\tilde{n}/dE = D(E)$.

Thus, at $T = 0$, we obtain for the conductivity exactly the Drude form, with $\tilde{n}(E_F)$ the electron density: $\sigma_{\mu\mu}(E) = e^2 \tau(E_F) \tilde{n}(E_F)/m$. 
B=0 conductivity for finite-range scatterers

We use the Ward identities $F^\sigma,\sigma (k) = \frac{1}{\hbar} \frac{\partial G_k^\sigma}{\partial \kappa_{\nu}} = \frac{\hbar k_{\nu}}{m} \frac{\partial G_k^\sigma}{\partial \varepsilon_k}$ and the notation $F^{+, -}(k) = \frac{\hbar k_{\nu}}{m} \Phi^{+, -}(k, E)$, take into account the diagonality of the conductivity tensor and write

$$\sigma_{\mu \mu}(E) = \frac{e^2 \hbar}{2\pi} g_s \int \frac{d^2 k}{(2\pi)^2} \frac{\hbar k_{\mu}}{m} \frac{\hbar k_{\mu}}{m} \left\{ \Phi^{+, -}(k, E) - \frac{1}{2} \frac{\partial}{\partial \varepsilon_k} [G_k^+ + G_k^-] \right\}$$

similar to the case of $\delta$-potential scatterers, where $\Phi^{+, -}(k, E) = G_k^+ G_k^-$. If we replace $\Phi^{+, -}(k, E)$ by $G_k^+ G_k^-$, we obtain, similar to the $\delta$-potential case,

$$\sigma^{(0)}_{\mu \mu}(E) = \frac{e^2 \hbar}{m} D_0 \left\{ \frac{1}{2\pi} + \int_0^\infty d\varepsilon_k \frac{E - \Delta_k(E)}{\Gamma_k(E)} A_k(E) \right\}.$$ 

Now the term in curly brackets can not be written as $n_{el}(E)/[\Gamma(E)D_0]$ to obtain the Drude form, since $\Gamma_k(E)$ depends on $\varepsilon_k$ and can not be taken out of the integral. To proceed we make the ansatz

$$\Phi^{+, -}(k; E) = 2\pi A_k(E) \phi_k(E),$$

with the spectral function $A_k(E) = \text{Im} G_k^-(E)/\pi = \Gamma_k(E) G_k^+(E) G_k^-(E)/2\pi$. Apparently, the replacement of $\Phi^{+, -}(k, E)$ by $G_k^+ G_k^-$ is equivalent with replacing $\phi_k(E)$ by $1/\Gamma_k(E)$, which is, however, only correct in the case of $\delta$-potential scatterers.
Boltzmann-like transport equation

The modified Bethe-Salpeter equation for $\Phi^{\sigma\sigma'}(k; E)$,

$$
\Phi^{+, -}(k; E) = G_k^+(E) \left\{ 1 + \Gamma_0 \int_0^\infty d\varepsilon_{k'} \int_{-\pi}^\pi \frac{d\chi}{2\pi} w(k, k'; \chi) \cos \frac{k'}{k} \Phi^{+, -}(k'; E) \right\} G_k^-(E),
$$

reads with $\Phi^{+, -}(k; E) = 2\pi A_k(E) \phi_k(E) = \Gamma_k(E) G_k^+(E) G_k^-(E) \phi_k(E)$:

$$
2\pi \Gamma_0 \int_0^\infty d\varepsilon_{k'} \int_{-\pi}^\pi \frac{d\chi}{2\pi} w(k, k'; \chi) A_{k'}(E) \left\{ \phi_k(E) - \cos \chi \frac{k'}{k} \phi_{k'}(E) \right\} = 1,
$$

where the term in front of the first curly bracket is just $\Gamma_k(E)$. If the normalized scattering cross section $w(k, k'; \chi)$ is independent of $\chi$ (as for $\delta$-scatterers), the $\cos \chi$ term averages to zero, and $\phi_k(E) = 1/\Gamma_k(E)$.

If we could neglect collision broadening effects on the spectral function, $A_k(E) \approx \delta(E - \varepsilon_k)$, we would need $\phi_k(E)$ (at $T = 0$) only for $k = k_E \equiv \sqrt{2mE/\hbar^2}$ at the Fermi energy $E = E_F$, and the energy integral could be performed to yield $\phi_{k_E}(E) = \tau_{tr}(E)/\hbar$ with

$$
\frac{\hbar}{\tau_{tr}(E)} = \Gamma_0 \int_{-\pi}^\pi d\chi w(k_E, k_E; \chi) \left[ 1 - \cos \chi \right],
$$

where $\tau_{tr}(E)$ has the meaning of the transport relaxation time of the classical Drude-Boltzmann theory. Without this approximation, $\phi_k(E)$ is not restricted to the energy shell $\varepsilon_k = E$, and must be calculated from the (one-variable) linear integral equation.
Correct conductivity formula
Finite range impurities, $B = 0$

Inserting the correct $\Phi^{+-}(k; E) = 2\pi A_k(E) \phi_k(E)$ means to insert into the integral of the preliminary formula for $\sigma^{(0)}_{\mu\mu}(E)$ an additional term $\varepsilon_k A_k(E)[\phi_k(E) - 1/\Gamma_k(E)]$. The result is

$$
\sigma_{\mu\mu}(E) = \frac{e^2 \hbar}{m} D_0 \left\{ \int_0^\infty d\varepsilon_k A_k(E) \left[ \varepsilon_k \phi_k(E) + \frac{E - \varepsilon_k - \Delta_k(E)}{\Gamma_k(E)} \right] + \frac{1}{2\pi} \right\}.
$$

The classical Drude formula $\sigma_{\mu\mu} = (e^2/m)\tau_{\text{tr}} n_{\text{el}}$ is recovered by

- neglecting collision broadening, $A_k(E) \to \delta(E - \varepsilon_k)$,
- neglecting the energy shift $\Delta_k(E)$, so that the second term in the square bracket vanishes,
- taking $E = E_F$ as the Fermi energy, so that $D_0 E_F = n_{\text{el}}$,
- neglecting the last term in the curly bracket as compared to $E_F \tau_{\text{tr}}/\hbar \gg 1$, where $\tau_{\text{tr}} = \hbar \phi_{k_F}(E_F)$.

For finite collision broadening it is, however, not justified to neglect the second term in the square bracket, since it yields a divergent contribution and cancels a divergence due to the first term.
Magnetoconductivities from short-range impurities

Diagonal components $\sigma_{\mu\mu}$

- For short-range, $\delta$-function like impurity potentials the selfenergy is a c-number and commutes with the position operator.
- Thus the Ward identities yield $\hat{F}^{\sigma,\sigma}(\hat{\nu}_\nu) = \hat{G}^\sigma \hat{\nu}_\nu \hat{G}^\sigma$, i.e., the vertex corrections vanish for $\sigma' = \sigma$.
- One can show that in this c-number limit the corresponding vertex corrections for the case $\sigma = -\sigma' = +$ vanish, too. Then: $\hat{F}^{\sigma\sigma'}(\hat{\nu}_\nu) = \hat{G}^\sigma \hat{\nu}_\nu \hat{G}^{\sigma'}$ and $\hat{D}(\hat{\nu}_\nu) = \frac{d\hat{G}}{dE} \hat{\nu}_\nu \hat{G} - \hat{G} \hat{\nu}_\nu \frac{d\hat{G}}{dE}$, respectively.
- Since the $\hat{D}(\hat{\nu}_\nu)$ do not contribute to the diagonal components of the conductivity tensor, these follow as

$$\sigma^{(l)}_{\mu\mu}(E) = \frac{e^2 \hbar}{2\pi F} \text{Sp} \hat{\nu}_\mu \left\{ \hat{G}^+ \hat{\nu}_\mu \hat{G}^- - \frac{1}{2} \left[ \hat{G}^- \hat{\nu}_\mu \hat{G}^- + \hat{G}^+ \hat{\nu}_\mu \hat{G}^+ \right]\right\}$$

$$= -\frac{i e^2 \hbar}{2F} \text{Sp} \hat{\nu}_\mu \left\{ \hat{A}(E) \hat{\nu}_\mu \hat{G}^-(E) - \hat{G}^+(E) \hat{\nu}_\mu \hat{A}(E) \right\},$$

$$= \frac{\pi e^2 \hbar}{F} \text{Sp} \hat{\nu}_\mu \hat{A}(E) \hat{\nu}_\mu \hat{A}(E),$$

in analogy to the $B = 0$ case.
Evaluation of $\sigma_{\mu\mu}$

Since the Green operator is diagonal in the Landau representation and independent of $k_y$, we use the notation $\langle n', k'_y | \hat{G}(z) | n, k_y \rangle = \delta_{n',n} \delta_{k'_y,k_y} G_n(z)$ with

$$G_n(z) = \left[ z - \hbar \omega_c (n + \frac{1}{2}) - \Sigma(z) \right]^{-1}, \quad \Sigma^\pm(E) = \Delta(E) \mp \frac{i}{2} \Gamma(E),$$

where $\Delta(E)$ and $\Gamma(E)$ are real functions of the energy, and $\Gamma(E)$ is non-negative. The spectral operator is also diagonal in the Landau representation and has the matrix elements $A_n(E) = \text{Im} G_n^{-}\langle E \rangle / \pi$. Performing the sum over $k_y$, we obtain from

$$\sigma_{\mu\mu}(E) = \frac{\pi e^2 \hbar}{2\pi \ell^2} \sum_{n,n'} \left| \langle n | \hat{v}_\mu | n' \rangle \right|^2 A_{n'} A_n = \omega_c e^2 \hbar \frac{\ell^2 \omega_c^2}{2} \sum_{n=0}^{\infty} (n + 1) A_{n+1} A_n,$$

with

$$\left| \langle n | \hat{v}_\mu | n' \rangle \right|^2 = \frac{\ell^2 \omega_c^2}{2} \left\{ (n + 1) \delta_{n',n+1} + n \delta_{n',n-1} \right\}$$

for both the $x$ and the $y$ component of the velocity, so that $\sigma_{yy} = \sigma_{xx}$.

Note that approximations with vanishing overlap of the spectral functions $A_n(E)$ of adjacent Landau levels, like the simple “quadratic-equation” approximation or the “jelly model”, yield vanishing conductivities $\sigma_{\mu\mu}$. 

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Peak values of $\sigma_{\mu\mu}(E)$

- For low temperatures and weak disorder, $k_b T \ll |\Sigma^-(E)| \ll \hbar\omega_c$, we can easily evaluate $\sigma_{\mu\mu}(E)$ to leading order in the small parameter $\Gamma(E)/\hbar\omega_c$.

- Assume that the Fermi energy is in the broadened Landau level with Landau quantum number $n = n_F$, so that $|E_F - \hbar\omega_c(n_F + 1/2)|$ is of the order of $\Gamma(E_F)$.

- Then the leading contribution comes from $n = n_F - 1$ and $n = n_F$,

$$\sigma_{\mu\mu}(E) \approx \omega_c^2 \frac{e^2 \hbar}{2} (2n_F + 1) A_{n_F}(E) \frac{\Gamma(E)}{2\pi(\hbar\omega_c)^2} = \frac{e^2}{\hbar} (n_F + \frac{1}{2}) \Gamma(E) A_{n_F}(E).$$

- Since the overlap of spectral functions is evaluated, one now may use the “quadratic equation approximation”: $A_n(E) \approx (\pi\Gamma_B)^{-1} \sqrt{1 - (E - \varepsilon_n)^2 / 4\Gamma_B^2}$ and $\Gamma(E) \approx 2\pi\Gamma_B^2 A_n(E)$ to estimate the peak values obtained for $E = \varepsilon_n = \hbar\omega_c(n + 1/2)$:

$$\sigma_{\mu\mu}(\varepsilon_n) \approx \frac{e^2}{\hbar} \frac{2n + 1}{\pi}$$

- In the Landau level with $n = n_F$, the product $\Gamma(E) A_{n_F}(E) \leq 2/\pi$ assumes values of order unity. The neglected contributions are by a factor $[\Gamma(E)/\hbar\omega_c]^2$ smaller.
Resummation of $\sigma_{\mu\mu}(E)$

We rewrite the product of adjacent spectral functions as

$$A_{n+1}A_n = (2\pi i)^{-2}(G_{n+1}^- - G_{n+1}^+)(G_n^- - G_n^+)$$

$$= \frac{1}{(2\pi i)^2} \left[ \frac{G_{n+1}^- - G_n^-}{\hbar \omega_c} - \frac{G_{n+1}^- - G_n^+}{\hbar \omega_c + i\Gamma} - \frac{G_{n+1}^+ - G_n^-}{\hbar \omega_c - i\Gamma} + \frac{G_{n+1}^+ - G_n^+}{\hbar \omega_c} \right]$$

$$= \frac{1}{(2\pi i)^2} \left\{ \frac{2}{\hbar \omega_c} - \frac{2\hbar \omega_c}{(\hbar \omega_c)^2 + \Gamma^2} \right\} \text{Re}(G_{n+1}^- - G_n^-) - \frac{2\Gamma}{(\hbar \omega_c)^2 + \Gamma^2} \text{Im}(G_{n+1}^- - G_n^+)$$

$$= \frac{2}{(2\pi i)^2 \hbar \omega_c} \frac{\Gamma}{(\hbar \omega_c)^2 + \Gamma^2} \left[ \text{Re}(G_{n+1}^- - G_n^-) - \hbar \omega_c \text{Im}(G_{n+1}^- - G_n^+) \right].$$

Now we rearrange and evaluate the relevant sum, so that all partial sums converge:

$$\sum_{n=0}^{\infty} (n + 1) \left[ \hbar \omega_c \left( \text{Im}G_{n+1}^- + \text{Im}G_n^- \right) - \Gamma \left( \text{Re}G_{n+1}^- - \text{Re}G_n^- \right) \right]$$

$$= \lim_{N \to \infty} \left\{ (N + 1) \left[ \hbar \omega_c \text{Im}G_{N+1}^- - \Gamma \text{Re}G_{N+1}^- \right] + \sum_{n=0}^{N} \left[ 2\varepsilon_n \text{Im}G_n^- + \Gamma \text{Re}G_n^- \right] \right\}$$

$$= \frac{\Gamma}{\hbar \omega_c} + \sum_{n=0}^{\infty} \frac{\varepsilon_n \Gamma + \Gamma(E - \varepsilon_n - \Delta)}{(E - \varepsilon_n - \Delta)^2 + (\Gamma/2)^2} = \frac{\Gamma}{\hbar \omega_c} + (E - \Delta) \sum_{n=0}^{\infty} 2\pi A_n.$$

This yields a result close to the Drude form

$$\sigma_{\mu\mu}(E) = \frac{e^2}{m} \frac{\hbar \Gamma(E)}{(\hbar \omega_c)^2 + [\Gamma(E)]^2} \tilde{n}(E),$$
Drude form of quantum conductivity $\sigma_{\mu\mu}(E)$

This yields at finite temperature the Drude form of the longitudinal conductivity:

$$\sigma_{\mu\mu} = \frac{e^2}{m} \int_{-\infty}^{\infty} dE \left( -\frac{df}{dE} \right) \frac{\tau(E) \tilde{n}(E)}{1 + [\omega_c \tau(E)]^2}, \quad \text{with} \quad \tau(E) = \frac{\hbar}{\Gamma(E)}.$$

Here we have (similar to the $B = 0$ case) introduced the definition

$$\tilde{n}(E) = [E - \Delta(E)] D(E) + D_0 \Gamma(E)/(2\pi),$$

where $D(E) = \mathcal{F}^{-1} \text{Sp}\hat{A}(E) = (2\pi \ell^2)^{-1} \sum_{n=0}^{\infty} A_n(E)$ is the density of states (DOS) of the disordered 2DES in the magnetic field $B$, and $D_0 = m/(2\pi \hbar^2)$ is the corresponding DOS in the absence of disorder and for $B = 0$. So far we have neglected the electron spin. For a spin-degenerate electron system we have to multiply $D(E)$, $D_0$, and therefore $\tilde{n}(E)$ and $\sigma_{\mu\mu}(E)$ by the spin degeneracy factor $g_s = 2$.

Note that here $\tilde{n}(E)$ is different from the integrated DOS and exhibits a peak structure due to the DOS factor, whereas the integrated DOS exhibits only a step structure.