QUESTIONS: Choose any four out of the following five problems

[25] 1. Show that the family of right circular cone \( x^2 + y^2 = c^2 z^2 \), where \( c \) is a parameter, form a equipotential surfaces, and show that the corresponding potential function is of the form \( \psi = A \log \tan(\theta/2) + B \), where \( A \) and \( B \) are constants and \( \theta \) is the usual angular coordinate in spherical coordinate system.

(This problem is from Sneddon, page 150 problem 2). Solution. Solution \( \psi \) of the Laplace equation is expressed as \( \psi = F(f) \). Surfaces \( f = \text{constant} \) are equipotential surfaces if

\[
\frac{F''}{F'} = \frac{-1}{f} - \frac{1}{2 \sqrt{1+f}}
\]

Then

\[
F = A \int f \frac{df}{\sqrt{1+f}} + B
\]

If we spherical coordinates then \( f = \tan^2 \theta \). Using this in the above integral we obtain the required result.

[25] 2. (a). Let \( \psi \) be the solution of the interior Dirichlet problem for some region \( V \) corresponding to the boundary value \( f \). Prove that the maximum and minimum values of \( \psi \) takes place on the boundary of \( V \). (b). If \( \psi_1, \psi_2 \) are solutions of the Dirichlet problem for some region \( V \) corresponding to boundary values \( f_1 \) and \( f_2 \) respectively, and if \( |f_1 - f_2| < \epsilon \) at all points of \( S \), prove that \( |\psi_1 - \psi_2| < \epsilon \) at all points of \( V \). (c). Prove that the solutions of the interior Dirichlet problem has unique solutions.

(This problem is from Sneddon Page 155 problem 1. Part (a) and Part (b) was solved in Class). Solution: (a). Solved in Class. (b). Let \( \psi = \psi_1 - \psi_2 \). Then \( \nabla^2 \psi = 0 \) in \( V \) and \( \psi = f = f_1 - f_2 \) on \( S \). Since \( \psi \) does not have any extremum values in \( V \), then \( |\psi| < \max |f| \) for all points in \( V \). Since \( \max |f| < \epsilon \) then \( |\psi| < \epsilon \) for all points in \( V \). (c) In the uniqueness problem \( f_1 = f_2 \), then \( f = 0 \). Hence \( |\psi| \leq 0 \) in \( V \). This leads to vanishing of \( \psi = 0 \).

(You could use the proof done in the class as well).

[25] 3. Prove that the solution \( \psi(r, \theta, \phi) \) of the the exterior Dirichlet problem for the unit sphere

\[
\nabla^2 \psi = 0, \quad r > 1,
\]

\[
\psi = f(\theta, \phi), \quad \text{on} \quad r = 1
\]
is given in terms of the solution \( v(r, \theta, \phi) \) of the interior Dirichlet problem
\[
\nabla^2 v = 0, \quad r < 1,
\]
\[
v = f(\theta, \phi), \quad \text{on } r = 1
\]
by the formula
\[
\psi(r, \theta, \phi) = \frac{1}{r} v\left( \frac{1}{r}, \theta, \phi \right)
\]
(This problem is from Sneddon Page 155 problem 4 and partly solved in Class). **Solution.** Interior and exterior BCs are the same at the spheres surface \( r = 1 \). We have to show that the function solves the Laplace equation. Since \( \psi(r, \theta, \phi) \) solves the Laplace equation then
\[
\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} L_{\theta,\phi} \psi = 0
\]
where \( L_{\theta,\phi} \) part of the Laplace operator is the angular part which does not effect our proof of the problem. Letting \( \psi(r, \theta, \phi) = \frac{1}{r} v(\xi, \theta, \phi) \) we find
\[
\psi_r = -\frac{1}{r^2} v(\xi, \theta, \phi) - \frac{1}{r^3} v_\xi(\xi, \theta, \phi), \quad (1)
\]
\[
r^2 \psi_r = -v(\xi, \theta, \phi) - \frac{1}{r} v_\xi(\xi, \theta, \phi), \quad (2)
\]
\[
(r^2 \psi_r)_r = \frac{2}{r^2} v_\xi + \frac{1}{r^3} v_{\xi,\xi} = \frac{1}{r} (\xi^2 v_\xi)_r \quad (3)
\]
Hence we obtain
\[
\nabla^2 \psi = \frac{\xi^2}{r^3} \nabla^2 \xi v(\xi, \theta, \phi) = 0
\]

[25]4. Find the potential function \( \psi(x, y, z) \) (satisfying the Laplace equation) in the region \( 0 \leq x \leq a, 0 \leq y \leq b, \) and \( 0 \leq z \leq c \) satisfying the following conditions
\[
\begin{align*}
(1) \quad & \psi = 0, \quad \text{on } x = 0, \ x = a, \ y = 0, \ y = b, \ z = 0, \\
(2) \quad & \psi = f(x, y) \quad \text{on } z = c, \ 0 \leq x \leq a, \ 0 \leq y \leq b
\end{align*}
\]
(This problem is solved by Sneddon Page 160 Example 6 and solved also in Class).

\[
\nabla^2 \psi = 0, \quad x > 0, \\
\psi(0, y, z) = f(y, z) \quad \text{and} \\
\psi \to 0 \quad \text{as } r \to 0
\]
Here $r^2 = x^2 + y^2 + z^2$. (b) If $f(y, z)$ does not depend on $z$, i.e., $f(y, z) = h(y)$ then simplify and interpret your solution. (This problem is solved by Sneddon and also in Class.)