MATH 584; TOPICS IN MATHEMATICAL PHYSICS II

Homework Set I

March 18, 2005

QUESTIONS:

Let \( f \in T^{(0,0)} M \), \( K \in T^{(1,0)} M \), \( \gamma \in T^{(0,1)} M \), \( \theta \in T^{(2,0)} M \), \( J \in T^{(0,2)} M \), \( \Phi \in T^{(1,1)} M \) where \( M \) is a smooth manifold.

1. Let \( \phi : M \to \tilde{M} \) be a \( C^\infty \) map. Let \( u \in M \) and \( \tilde{u} \in \tilde{M} \). Prove that
   (i) \( \tilde{f}(\tilde{u}) = f(u) \).
   (ii) \( \tilde{K}(\tilde{u}) = \phi'(u) K(u) \).
   (iii) \( \tilde{\gamma}(u) = \phi'^\dagger(u) \tilde{\gamma}(\tilde{u}) \).
   (iv) \( \tilde{\theta}(\tilde{u}) = \phi'(u) \theta(u) \phi'^\dagger(u) \).
   (v) \( \tilde{J}(u) = \phi'^\dagger(u) \tilde{J}(\tilde{u}) \phi'(u) \).
   (vi) \( \phi'(u) \Phi(u) = \tilde{\Phi}(\tilde{u}) \phi'(u) \).

2. Prove that the Lie derivatives of these tensors with respect to the vector filed \( v \) are as follows:
   (i) \( L_v f = f' \circ v = \langle \nabla f, v \rangle \).
   (ii) \( L_v K = K' \circ v - v' \circ K = [v, K] \).
   (iii) \( L_v \gamma = \gamma' \circ v + v'^\dagger \circ \gamma \).
   (iv) \( L_v \Phi = \Phi' \circ v + \Phi \circ v' - v' \circ \Phi \).
   (v) \( L_v \theta = \theta' \circ v - \theta \circ v'^\dagger - v' \circ \theta \).
   (vi) \( L_v J = J' \circ v + J \circ v' + v'^\dagger \circ J \).
   (vii) \( L_v \Psi = \Psi' \circ v - \Psi \circ v'^\dagger + v'^\dagger \circ \Psi \). Here \( \Psi = \Phi^\dagger \)

3. Consider an evolution equation
   \[ u_t = K(u) \]  \hspace{1cm} (1)
   where \( K \in T^{(1,0)} M \). We assume that \( u \) and its derivatives go to zero as \( |x| \to \infty \).
(i). A vector field $\sigma \in TM$ is a \textit{strong symmetry} if and only if
\begin{equation}
\frac{\partial \sigma}{\partial t} + \sigma'[K] - K'[\sigma] = 0.
\end{equation}

(ii). A co-vector $\gamma \in T^*M$ is \textit{conserved gradient} (gradient of a functional $I$) if and only if
\begin{equation}
\frac{\partial \gamma}{\partial t} + \gamma'[K] + K'[\gamma] = 0.
\end{equation}
and $\gamma' = \gamma'^t$. Co-vectors satisfying only (3) are called \textit{conserved covariants}.

(iii). An operator valued function $\Phi \in T^{(1,1)}M$ is \textit{recursion operator} of (1) (or it is a \textit{strong symmetry} of $K$) if and only if the following operator equation is valid.
\begin{equation}
\Phi'[K] - K'[\Phi] + \Phi K' = 0.
\end{equation}

(iv). An operator valued function $\Phi \in T^{(1,1)}M$ is called \textit{hereditary} (or \textit{Nijenhuis} or \textit{regular}) if and only if
\begin{equation}
\Phi'[\Phi v]w - \Phi \Phi'[v]w = \Phi'[\Phi w]v - \Phi \Phi'[w]v
\end{equation}
for all vector fields $v, w \in TM$.

(v). An operator valued function $\theta \in T^{(2,0)}M$ is called a \textit{Hamiltonian operator} if and only if it is skew symmetric, $\theta = -\theta^t$, and satisfies
\begin{equation}
<a, \theta'[\theta b]c>_+\text{cyclic permutations} = 0
\end{equation}
for all $a, b, c \in T^*M$.

(vi). An equation (1) is a \textit{Hamiltonian system} if and only if it can be written in the form
\begin{equation}
u_t = \theta \gamma
\end{equation}
where $\theta$ is a Hamiltonian operator and $\gamma$ is a closed co-vector ($\gamma = \text{grad } I$ which implies $\gamma' = \gamma'^t$).

(vii). Associated with (7) we define the following Poisson bracket:
\begin{equation}
\{I, H\} = <\text{grad } I, \theta \text{grad } H>
\end{equation}
Try to write the above equations in the form of Lie derivatives and give explanations.

4. Let $K(u) = u_{xxx} + 6uu_x$.
   (i). Find $K'$ and $K''$. 
   (ii). Let $I = \int_R (u^3 - \frac{1}{2} u_x^2) \, dx$ Find the co-vector $\gamma = \text{grad } I$. Find $\gamma'$ and show that $\gamma' = \gamma''$. Show that $\gamma$ is a conserved gradient.
   (iii). Let $\sigma = 1 + 6tu_x$. Show that $\sigma$ is a symmetry of the KdV equation (1).
   (iv). Show that the KdV equation is a Hamiltonian system by writing (1) in the form (7) where
   \[
   \theta = D, \quad \gamma = u_{xx} + 3u^2 
   \] (9)
   Show that $\theta$ given above is a Hamiltonian operator and $\gamma$ given above is conserved gradient.
   (v). KdV equation has also a second Hamilton operator.
   \[
   \theta = D^3 + 4uD + 2u_x, \quad \gamma = u 
   \] (10)
   where $u_t = (D^3 + 4uD + 2u_x) u$. Call the previous Hamilton operator as $\theta_0 = D$ and conserved gradient as $\gamma_1$. For the new pair call $\theta_1$ and $\gamma_0$ respectively. Prove that $\theta_1$ is a Hamilton operator and $\gamma_0$ is a conserved covariant.
   (vi). Prove that the operator $\Phi = D^2 + 4u + 2u_x D^{-1}$ is a recursion operator of the KdV equation. Here $D^{-1} f = \int_{-\infty}^{\infty} f(\xi) \, d\xi$
   (vii). Prove that $\Phi$ is hereditary.
   (viii). Show that $\Phi = \theta_1 \theta_0^{-1}$.
   (ix). Using the recursion operator $\Phi \in T(1,1)M$ find the first four symmetries of the KdV equation
    \[
    \sigma_{n+1} = \Phi \sigma_n, \quad n = 0, 1, 2, 3, 4 \quad (\sigma_0 = u_x)
    \]
   (x). The adjoint $\Phi^\dagger$ of $\Phi$ maps conserved gradients to conserved gradients ($\Phi^\dagger : T^*M \to T^*M$)
   \[
   \gamma_{n+1} = \Phi^\dagger \gamma_n, \quad n = 0, 1, 2, \ldots
   \]
Find first four conserved gradients of the KdV by taking $\gamma_0 = u$. (xi) Prove the following: Hamilton operators map conserved gradients $\gamma_n$ to symmetries $\sigma_n$ ($\theta : T^*M \to TM$)

$$\sigma_n = \theta \gamma_n, \quad n = 0, 1, 2, \cdots$$