HOMEWORK II (MATH 544): MARCH 16, 2006

Higher order equations with one spatial variable

1. Let

\[ u_{tt} + a^2 u_{xxxx} = 0, \quad 0 < x < L, \quad t > 0, \]

\[ u(x, 0) = f(x), \quad u_t(x, 0) = 0, \quad 0 \leq x \leq L, \quad t \geq 0 \]

\[ u(0, t) = u_x(0, t) = 0, \quad u(L, t) = u_{xx}(L, t) = 0, \quad t \geq 0 \]

(a). Find the formal solution of the above IVP.
(b). Find reasonable restrictions on \( f(x) \) so that the above formal solution in part (a) is a solution of the IVP.

(c) Prove the uniqueness of this IVP. [Hint: Consider the energy functional as

\[ E(t) = \frac{1}{2} \int_0^L \left[ (u_t)^2 + a^2 (u_{xx})^2 \right] dx \]

(d). Prove that the solution in part (b) continuously depends on the initial data.

2. Consider the IVP defined on the whole real line

\[ u_{tt} + a^2 u_{xxxx} = 0, \quad -\infty < x < \infty, \quad t > 0 \]

\[ u(x, 0) = f(x), \quad u_t(x, 0) = g(x), \quad -\infty < x < \infty \]

Use Fourier transform

\[ u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-iwx} \hat{u}(w, t) \, dw \]

to solve this problem. Find the restrictions on the data \((f, g)\) so that the above IVP has a unique solution

3. Linearized version of the Korteweg de Vries equation. Let \( u_t = \sigma u_{xx} \), \( \sigma > 0 \) (a constant) Use Fourier Transform to solve this equation in \( -\infty < x < \infty, \quad t > 0 \) and subject to the initial condition \( u(x, 0) = f(x) \). Show that the solution will be of the form

\[ u(x, t) = \int_{-\infty}^{\infty} k(x - y, \sigma t) f(y) \, dy \]
where
\[ k(x, t) = \frac{1}{(3t)^{1/3}} Ai\left(-\frac{x}{(3t)^{1/3}}\right) \]
and \(Ai(\lambda)\) is the Airy function.

4. Show that there exists at most one solution of the BVP
\[
\begin{align*}
  u_t &= \sigma u_{xxx}, \quad 0 < x < L, \quad t > 0 \\
  u(x, 0) &= f(x), \quad 0 \leq x \leq L \\
  u(0, t) &= u_x(0, t) = 0, \quad u(L, t) = 0, \quad t \geq 0
\end{align*}
\]
and solve it by means of separation of variables. \([Hint: \ Use \ the \ energy \ functional \ as \ E(t) = \frac{1}{2} \int_0^L u^2 dx]\)

5. Show that the Riemann-Green function for
\[
\frac{\partial^2 u}{\partial x \partial y} + \frac{2}{x+y} \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) = 0
\]
is
\[
R(x, y, \xi, \eta) = \frac{x+y}{(\xi + \eta)^3} \left[(x+y)(\xi + \eta) + 2(x - \xi)(y - \eta)\right]
\]
Use the Riemann method to show that the solution which satisfies the conditions \(u = 0, u_x = 3x^2\) on \(y = x\) is
\[
u(x, y) = 2x^3 - 3x^2y + 3xy^2 - y^3\]