THE METHOD OF Riemann
Ronald B Guenther and John W Lee, Partial Differential Equations of Mathematical Physics and Integral Equations

1. Solve the following initial value problem

\[ u_{xy} - yu_x = 1, \quad y > -x, \quad -\infty \leq x \leq \infty, \]  
\[ u(x, -x) = x, \quad u_x(x, -x) = x^2, \quad u_y(x, -x) = x^2 - 1, \]  
\[ \Gamma = \{x, y \in \mathbb{R}^2 | y = -x\} \]

**step 1.** Define the Riemann function \( v(x, y) \) which satisfies the homogeneous adjoint equation

\[ v_{xy} + (yv)_x = 0 \]

Using the equations for \( u \) and \( v \) we obtain

\[ A_y + B_x = v \]

where

\[ A = \frac{1}{2}(vu_x - w_y), \quad B = \frac{1}{2}(vu_y - w_y) - yuv \]

**step 2.** Integrate the above divergence form in a triangular region with the vertices \( M(x_0, y_0), P(-y_0, y_0), \) and \( Q(x_0, -x_0) \) and using Green's theorem we get

\[ \int_C [-Adx + Bdy] = \int \int_{T(x_0,y_0)} v(x,y)dxdy \]

where \( C = L_{QM}uL_{MP}uL_{PQ} \). Evaluating each integral

\[ \int_{QM} Bdy = \frac{1}{2} \int_{QM} (vu_y - w_y)|_{x=x_0}dy - \int_{QM} yuv|_{x=x_0}dy \]

\[ = \frac{1}{2}(uv)|_Q^M - \int_{QM} u(x_0, y) (v_y + yv)dy \]
Since the last term contains an unknown value $u(x_0, y)$ of the function $u(x, y)$ we then let its coefficient to vanish.

$$v_y(x_0, y) + yv(x_0, y) = 0, \quad y \leq y_0 \quad (4)$$

Hence

$$\int_{Q_M} = \frac{1}{2}u(M)v(M) - \frac{1}{2}u(Q)v(Q)$$

Similarly we find that

$$\int_{M_P} = -\int_{M_P} A|_{y=y_0} dx = \frac{1}{2}u(M)v(M) - \frac{1}{2}u(P)v(P)$$

with

$$v_x(x, y_0) = 0, \quad x \leq x_0 \quad (5)$$

We then find that

$$u(M)v(M) = \frac{1}{2}u(P)v(P) + \frac{1}{2}u(Q)v(Q) - \int_{T}[-Adx+Bdy] + \int_{T(x_0,y_0)} v(x,y) dxdy \quad (6)$$

Here

$$u(M) = u(x_0, y_0), \quad v(M) = v(x_0, y_0),$$

$$u(P) = u(-y_0, y_0) = y_0, \quad v(P) = v(-y_0, y_0)$$

$$u(Q) = u(x_0, -x_0) = x_0, \quad v(Q) = v(x_0, -x_0),$$

**step 3.** Determination of the Riemann (Green) function: Let us collect all the conditions on the (Riemann) function $v(x, y)$. We usually use the notation $v(x, y) = R(x_0, y_0, x, y)$

$$v_{xy}(x, y) + yv(x, y) = 0, \quad y > -x, x \in R, \quad (7)$$

$$v_y(x_0, y) + yv(x_0, y) = 0, \quad y \leq y_0, \quad (8)$$

$$v_x(x, y_0) = 0, \quad x \leq x_0, \quad (9)$$

$$v(x_0, y_0) = 1. \quad (10)$$
The complete solution of the above problem (known as the Goursat-Problem) is easy and given by

\[ v(x, y) = e^{\frac{1}{2}(y^2 + x_0^2)} \]

**step 4.** Solution: Inserting the above result in the equation (6) we obtain that

\[
\begin{align*}
    u(x_0, y_0) &= -\frac{1}{2}y_0 + \frac{1}{2}x_0 e^{(y_0^2 - x_0^2)/2} + \frac{3}{2} \int_{-x_0}^{x_0} x^2 e^{(y_0^2 - x^2)/2} dx \\
    &= -\frac{1}{2}y_0 + \frac{1}{2}x_0 e^{(y_0^2 - x_0^2)/2} + \frac{3}{2} \int_{-y_0}^{y_0} e^{(y_0^2 - y^2)/2} dy + \int_{T(x_0, y_0)} e^{(y_0^2 - y^2)/2} dxdy,
\end{align*}
\]

\[
\begin{align*}
    &= -\frac{1}{2}y_0 + \frac{1}{2}x_0 e^{(y_0^2 - x_0^2)/2} + \int_{-y_0}^{x_0} \left[ \frac{3}{2} \xi^2 - \frac{1}{2} x_0 - \xi \right] e^{(y_0^2 - \xi^2)/2} d\xi
\end{align*}
\]

2. Solve the above problem without using Riemann’s method.

3. Solve the initial value problem by the use of Riemann’s method:

\[
\begin{align*}
    u_{xy} &= F(x, y), \quad u(x, x) = f(x), \quad u_x(x, x) = g(x)
\end{align*}
\]

4. Solve the following initial value problem:

\[
\begin{align*}
    u_{xy} + xu_x &= 1, \quad y > x, \quad x \in \mathbb{R}, \\
    u(x, -x) &= u_x(x, -x) = 0.
\end{align*}
\]

5. Solve

\[
\begin{align*}
    u_{xy} - u_x &= xy, \\
    u(x, x^3) &= \sin x, \quad u_x(x, x^3) = \cos x, \quad u_y(x, x^3) = 0,
\end{align*}
\]

in the region above the curve \( \Gamma : y = x^3 \).