Traveling wave solutions of degenerate coupled Korteweg-de Vries equation

Metin Gürses1,a) and Aslı Pekcan2,b)

1Department of Mathematics, Faculty of Science, Bilkent University, 06800 Ankara, Turkey
2Department of Mathematics, Faculty of Science, Hacettepe University, 06800 Ankara, Turkey

(Received 21 December 2013; accepted 4 August 2014; published online 2 September 2014)

We give a detailed study of the traveling wave solutions of (\(\ell = 2\)) Kaup-Boussinesq type of coupled KdV equations. Depending upon the zeros of a fourth degree polynomial, we have cases where there exist no nontrivial real solutions, cases where asymptotically decaying to a constant solitary wave solutions, and cases where there are periodic solutions. All such possible solutions are given explicitly in the form of Jacobi elliptic functions. Graphs of some exact solutions in solitary wave and periodic shapes are exhibited. Extension of our study to the cases \(\ell = 3\) and \(\ell = 4\) are also mentioned. © 2014 AIP Publishing LLC. [http://dx.doi.org/10.1063/1.4893636]

I. INTRODUCTION

Multi-component Kaup-Boussinesq (KB) equations can be obtained from the Lax operator

\[
L = D^2 - \sum_{k=1}^{\ell} \lambda^{k-1} q^k(x, t),
\]

where \(q^k(x, t), k = 1, 2, ..., \ell\) are the multi-KB fields.1–4 Here \(\ell \geq 2\) is a positive integer.

The multi-system of KB equation is given as

\[
\begin{align*}
    u_t &= \frac{3}{2} uu_x + q^2_x, \\
    q^1_x &= q^1 u_x + \frac{1}{2} uq^2_x + q^3_x, \\
    &\vdots \\
    q^{\ell-1}_x &= q^{\ell-1} u_x + \frac{1}{2} uq^{\ell-1}_x + v_x, \\
    v_t &= -\frac{1}{4} uu_{xxx} + vu_x + \frac{1}{2} uv_x,
\end{align*}
\]

where \(q^1 = u\) and \(q^\ell = v\). This system in (2) was shown to be also a degenerate Korteweg-de Vries (KdV) system of rank one.5–7 This system admits also recursion operator for all values of \(\ell\). In this work, we shall investigate the traveling wave solutions of these coupled equations. For this purpose we start with the case \(\ell = 2\). To find such solutions we use time and space translation symmetries of the coupled system.

a)gurses@fen.bilkent.edu.tr
b)aslipekcan@hacettepe.edu.tr
The KB equation for $\ell = 2$ is

\[
\begin{align*}
  u_t &= \frac{3}{2} uu_x + v_x \\
  v_t &= -\frac{1}{4} u_{xxx} + uu_x + \frac{1}{2} uv_x.
\end{align*}
\]

(3)

In Ref. 8, the inverse problem of the above system was studied and soliton solutions which decay asymptotically were found. The $N = 1$ solution found in that work corresponds to the interaction of two solitary waves. It was also mentioned in Ref. 8 that there is no solution in the form of traveling wave solutions of system of equations for $\ell = 2$. This is consistent with the observation of Ref. 8.

We show that this is also valid for $\ell = 4$. We claim it to be true for all even positive integers. We show that it is possible to find solitary wave solutions of (3) which asymptotically decay to non-zero constants. Furthermore, in addition to the solitary wave solutions of (3) we find all traveling wave solutions which are expressible in terms of Jacobi elliptic functions.

Traveling wave solutions of a system of equations can be obtained if the equations possess time and space translation symmetries. Such symmetries exist in our case. Hence letting $x - ct = \xi$, where $c$ is a constant (the speed of the wave) and $u(x, t) = f(\xi)$, and $v(x, t) = g(\xi)$ from the first equation of (3) we have

\[ -cf'' = \frac{3}{2} ff' + g', \]

which gives

\[ g(\xi) = -cf - \frac{3}{4} f^2 + d_1, \]

(4)

where $d_1$ is an integration constant. Using $g(\xi)$ in the second equation of (3) yields

\[ -\frac{1}{4} f''' - 3cf f' - \frac{3}{2} f^2 f' + (d_1 - c^2) f' = 0. \]

Integrating above equation once we obtain

\[ -\frac{1}{4} f'' - \frac{3}{2} c f^2 - \frac{1}{2} f^3 + (d_1 - c^2) f + d_2 = 0. \]

By using $f'$ as an integrating factor, we can integrate once more. Finally, we get

\[ (f')^2 = -f^4 - 4cf^3 + 4(d_1 - c^2) f^2 + 8d_2 f + 8d_3 = F(f), \]

(5)

where $c$, $d_1$, $d_2$, $d_3$ are constants. These constants can be determined from the initial conditions $f(0)$, $f'(0)$, $f''(0)$, and $g(0)$. If $F(f)$ has zeros, these zeros are related to these initial conditions. For asymptotically decaying solutions of ($\ell = 2$) KB equations $f$, $f'$, $f''$, $f'''$, $g$, and $g'$ go to zero as $\xi \to \pm \infty$. Here in this work we shall find all possible solutions $f$ of (5). Given a solution $f$ one can find the corresponding solution $g(\xi)$ from (4).

In Refs. 9 and 10, a KB like system

\[
\begin{align*}
  h_t + (uh)_x + \frac{1}{4} u_{xxx} &= 0, \\
  u_t + uu_x + h_x &= 0,
\end{align*}
\]

(6)

was considered. Traveling wave solutions of this system satisfy a differential equation like (5) but the corresponding polynomial $F_1(f)$ is asymptotically positive definite. This means that the above KB like system possesses asymptotically decaying traveling wave solutions. In Refs. 9 and 10, some solitary wave solutions were found. The fourth degree polynomial arising in traveling wave solutions of the system (6) is different than the one given in (5). Hence the behavior of solutions here in this work and in Refs. 9 and 10 are different.
In Ref. 11, a modified version of the system (6), i.e.,
\begin{align*}
    h_t + (uh)_x + \frac{1}{4} \varepsilon^2 u_{xxx} &= 0, \\
    u_t + uu_x + h_x &= 0,
\end{align*}
(7)
was considered, where \( \varepsilon \) is a parameter which controls the dispersion effects. The upper sign is for the case when the gravity force dominates over the capillary one, and the lower sign is for the opposite case when capillary dominates over the gravity. The traveling wave solutions of the above system (7) were considered in Ref. 11. Equation (5) becomes now \( \varepsilon^2 (f')^2 = \pm F_2(f) \). In both cases, solitary wave solutions (dark and bright solitons) were found in Ref. 11. The lower case (negative sign) resembles to our case. Hence our solution in Sec. III A can be considered as a dark soliton in the sense of Ref. 11. This is the solution corresponding to one double and two simple zeros of the polynomial \( F(f) \). We have all other solutions corresponding to different combinations of the zeros of \( F(f) \) in Secs. III–V.

The layout of our paper is as follows: In Sec. II, we study the behavior of the solutions in the neighborhood of the zeros of \( F(f) \) and discuss all possible cases. We find all solitary wave solutions of the system (3) in Sec. III. These correspond to one double and two simple zeros of \( F(f) \), and one triple and one simple zeros of \( F(f) \). In Sec. IV, we find all elliptic type of solutions starting from very special ones to the most general elliptic type of solutions. These solutions are given in terms of the zeros of the function \( F(f) \). In Sec. V, we discuss \( \ell = 3 \) and \( \ell = 4 \) cases. In Sec. VI, we give the graphs of the solutions corresponding to all cases considered in the text.

II. GENERAL WAVES OF PERMANENT FORM FOR \( (\ell = 2) \)

Proposition 2.1. There is no real asymptotically vanishing traveling wave solution of Eq. (3) in the form \( u(x, t) = f(\xi) \) and \( v(x, t) = g(\xi) \), where \( \xi = x - ct \).

Proof. If we apply the boundary conditions \( f, f', f'', f''' \), \( g, g' \to 0 \) as \( \xi \to \pm \infty \) which describe the solitary wave, we get \( d_1 = d_2 = d_3 = 0 \). Hence we end up with
\begin{align*}
    (f')^2 = -f^4 - 4cf^3 - 4c^2 f^2 &= -f^2(f^2 + 4cf + 4c^2) \\
    &= -f^2(f + 2c)^2.
\end{align*}
Clearly, we do not have a real solution \( f \).

Now we will deal with Eq. (5). In order to have real solutions, \( d_1, d_2, d_3 \) must take values so that the following inequality holds:
\[ 4d_1 f^2 + 8d_2 f + 8d_3 \geq f^2(f + 2c)^2. \]

A. Zeros of \( F(f) \) and types of solutions

Here we will analyze the zeros of \( F(f) \).

(i) If \( f_1 = f(\xi_1) \) is a simple zero of \( F(f) \) we have \( F(f_1) = 0 \). Taylor expansion of \( F(f) \) gives
\begin{align*}
    (f')^2 = F(f) = F(f_1) + F'(f_1)(f - f_1) + O((f - f_1)^2) \\
    &= F'(f_1)(f - f_1) + O((f - f_1)^2).
\end{align*}
From here we get \( f'(\xi_1) = 0 \) and \( f''(\xi_1) = F'(f_1)/2 \). Hence we can write the function \( f(\xi) \) as
\begin{align*}
    f(\xi) &= f(\xi_1) + (\xi - \xi_1)f'(\xi_1) + \frac{1}{2}(\xi - \xi_1)^2 f''(\xi_1) + O((\xi - \xi_1)^3) \\
    &= f_1 + \frac{1}{4}(\xi - \xi_1)^2 F'(f_1) + O((\xi - \xi_1)^3). \quad (8)
\end{align*}
Thus, in the neighborhood of \( \xi = \xi_1 \), the function \( f(\xi) \) has local minimum or maximum as \( F'(f_1) \) is positive or negative, respectively, since \( f''(\xi_1) = F'(f_1)/2 \).

(ii) If \( f_1 = f(\xi_1) \) is a double zero of \( F(f) \) we have \( F(f_1) = F'(f_1) = 0 \). Taylor expansion of \( F(f) \) gives

\[
(f')^2 = F(f) = F(f_1) + F'(f_1)(f - f_1) + \frac{1}{2}(f - f_1)^2 F''(f_1) + O((f - f_1)^3)
\]

\[
= \frac{1}{2}(f - f_1)^2 F''(f_1) + O((f - f_1)^3).
\]

To have real solution \( f \), we should have \( F''(f_1) > 0 \). From the equality (9) we get

\[
f' \pm \frac{1}{\sqrt{2}} f \sqrt{F''(f_1)} \sim \pm \frac{1}{\sqrt{2}} f_1 \sqrt{F''(f_1)},
\]

which gives

\[
f(\xi) \sim f_1 + \alpha e^{\pm \frac{1}{\sqrt{2}} \sqrt{F''(f_1)} \xi},
\]

where \( \alpha \) is a constant. Hence \( f \to f_1 \) as \( \xi \to \pm \infty \). The solution \( f \) can have only one peak and the wave extends from \( -\infty \) to \( \infty \).

(iii) If \( f_1 = f(\xi_1) \) is a triple zero of \( F(f) \) we have \( F(f_1) = F'(f_1) = F''(f_1) = 0 \). Taylor expansion of \( F(f) \) gives

\[
(f')^2 = F(f) = F(f_1) + F'(f_1)(f - f_1) + \frac{1}{2}(f - f_1)^2 F''(f_1) + \frac{1}{6}(f - f_1)^3 + O((f - f_1)^4)
\]

\[
= \frac{1}{6}(f - f_1)^3 F''(f_1) + O((f - f_1)^4).
\]

This is valid only if both signs of \( (f - f_1)^3 \) and \( F'''(f_1) \) are same, i.e., we have the following two possibilities to have real solution \( f \):

1. \( (f - f_1) > 0 \) and \( F'''(f_1) > 0 \),
2. \( (f - f_1) < 0 \) and \( F'''(f_1) < 0 \).

Let us analyze these cases. If \( (f - f_1) > 0 \) and \( F'''(f_1) > 0 \), then we have

\[
f' \sim \pm \frac{1}{\sqrt{6}} (f - f_1)^{3/2} \sqrt{F'''(f_1)},
\]

which gives

\[
f(\xi) \sim f_1 + \frac{4}{(\pm \frac{1}{\sqrt{6}} \sqrt{F'''(f_1)} \xi + \alpha_1)^2},
\]

where \( \alpha_1 \) is a constant. Thus \( f \to f_1 \) as \( \xi \to \pm \infty \) if \( F'''(f_1) > 0 \).

Let \( (f - f_1) < 0 \) and \( F'''(f_1) < 0 \) hold. In this case, \( (f_1 - f) > 0 \) and \( F'''(f_1) = -G(f_1), G(f_1) > 0 \).

Then

\[
f' \sim \pm \frac{1}{\sqrt{6}} (f_1 - f)^{3/2} \sqrt{G(f_1)},
\]

which yields

\[
f(\xi) \sim f_1 - \frac{4}{(\pm \frac{1}{\sqrt{6}} \sqrt{G(f_1)} \xi + \alpha_2)^2},
\]

where \( \alpha_2 \) is a constant. Thus \( f \to f_1 \) as \( \xi \to \pm \infty \) if \( F'''(f_1) = -G(f_1) < 0 \).

(iv) If \( f_1 = f(\xi_1) \) is a quadruple zero of \( F(f) \), then there is only one possibility \( F(f) \)

\[
= -(f - f_1)^4 = (f')^2.
\]

It is clear that this case does not give a real solution except when \( f = f_1 \).
B. All possible cases

Here we present the sketches of the graphs of $F(f)$. Real solutions $(f')^2 = F(f) \geq 0$ occur in the shaded regions.

Now we analyze all possible cases about the zeros of $F(f)$ and above graphs in Fig. 1.

1. No real zero. If there is no real zeros of $F(f)$, then $F(f) < 0$. Hence there is no real solution of (5) in that case.

2. Two simple real zeros. If there is a simple zero $f_1$ of $F(f)$, since the order of $F(f)$ is four, there should be another simple zero $f_2$ of $F(f)$. The corresponding graph to this case is given in (a). Here, the real solution occurs when $f$ is between two different simple zeros $f_1$ and $f_2$. At $f_1$, $F'(f_1) = f''(\xi_1) > 0$ so graph of the function $f$ is concave up at $\xi_1$. At $f_2$, $F'(f_2) = f''(\xi_2) > 0$ hence graph of the function $f$ is concave up at $\xi_2$. Thus it is clear that the solution is periodic.

3. One double zero. If there is only one double zero $f_1$, then

$$(f')^2 = -(f - f_1)^2(f^2 + pf + q),$$ (14)

where $f^2 + pf + q$ has no real zero. This means $p^2 - 4q < 0$ which yields that $f^2 + pf + q > 0$. Then $(f')^2 = -(f - f_1)^2(f^2 + pf + q) < 0$ hence there is no real solution in that case except when $f = f_1$. Similarly, in the case when $F(f)$ has two double zeros $f_1$ and $f_2$, no real solutions exist since $(f')^2 = -(f - f_1)^2(f^2 - f_2)^2 < 0$ except when $f = f_1$ or $f = f_2$.

4. One double and two simple zeros. The corresponding graphs for this case are (b), (c), and (d). In (b) and (c), there are two different simple zeros $f_1$ and $f_3$ and one double zero $f_2$. We have $f_2 < f_1 < f_3$ in (b) and in the graph (c), $f_1 < f_3 < f_2$. In both cases, the real solution occurs when $f$ is between two simple zeros $f_1$ and $f_3$. At $f_1$, $F'(f_1) = f''(\xi_1) > 0$ so graph of the function $f$ is concave up at $\xi_1$. At $f_3$, $F'(f_3) = f''(\xi_3) > 0$ hence graph of the function $f$ is concave up at $\xi_3$. It is clear that the solution is periodic in this case.

In (d), different than the graphs (b) and (c) we have $f_1 < f_2 < f_3$. The real solution occurs when $f$ stays between $f_1$ and $f_2$ or $f_2$ and $f_3$. At $f_1$, $F'(f_1) = f''(\xi_1) > 0$ hence graph of the function $f$ is concave up at $\xi_1$. At double zero $f_2$, $f \to f_2$ as $\xi \to \pm\infty$. Hence we have a solitary wave solution with amplitude $f_1 - f_2 < 0$.

Similarly at $f_3$, $F'(f_3) = f''(\xi_3) < 0$, hence graph of the function $f$ is concave down at $\xi_3$. Therefore, we also have a solitary wave solution with amplitude $f_3 - f_2 > 0$. Explicit solitary wave solution for this case can be found in Sec. III.

FIG. 1. All possible sketches of the graphs of $F(f)$. 
(5) One triple and one simple zero. For this case, we can analyze the graphs (e) and (f). In (e), \( f_1 \) is simple and \( f_2 \) is triple zeros of \( F(f) \). We see that \( F''(f_1) = f''(\xi_1) > 0 \) hence graph of the function \( f \) is concave up at \( \xi_1 \). From the case (iii), we know that \( f \to f_2 \) as \( \xi \to \pm\infty \) for \( f - f_2 < 0 \) and \( F''(f_2) < 0 \). Hence we have solitary wave solution with amplitude \( f_1 - f_2 < 0 \).

Similarly, in (f) we have one triple zero \( f_1 \) and one simple zero \( f_2 \). For triple zero \( f_1 \) we have \( f \to f_1 \) as \( \xi \to \pm\infty \) for \( f - f_1 > 0 \) and \( F''(f_1) > 0 \). For simple zero we have \( F''(f_2) = f''(\xi_2) < 0 \) therefore graph of the function \( f \) is concave down at \( \xi_2 \). Clearly, we have a solitary wave solution with amplitude \( f_2 - f_1 > 0 \). Explicit solitary wave solution for this case can be found in Sec. III.

(6) Four different simple zeros. The corresponding graph for this case is given in (g). Here, there are four simple zeros \( f_1 < f_2 < f_3 < f_4 \). For \( f_1 \) and \( f_3 \), we have \( F'(f_1) = f''(\xi_1) > 0 \) and \( F'(f_3) = f''(\xi_3) > 0 \) thus graph of the function \( f \) is concave up at \( \xi_1 \) and \( \xi_3 \). For \( f_2 \) and \( f_4 \), we have \( F'(f_2) = f''(\xi_2) < 0 \) and \( F'(f_4) = f''(\xi_4) < 0 \) so graph of the function \( f \) is concave down at \( \xi_2 \) and \( \xi_4 \). Obviously, the solution is periodic.

As a summary we have the following results. By solution below, we mean non-constant solutions.

Proposition 2.2. Equation (5) has no real solutions when the function \( F(f) \) has one of the following properties: (i) it has no real zeros, (ii) it has only two real zeros, (iii) it has only one double zero, (iv) it has only two double zeros, and (v) it has a quadruple zero.

Proposition 2.3. Equation (5) admits solitary wave solutions when the function \( F(f) \) admits (i) one double and two simple zeros and (ii) one triple and one simple zeros.

From Proposition 2.2, we can conclude that the function \( F(f) \) must have four zeros,

\[ F(f) = -(f - f_1)(f - f_2)(f - f_3)(f - f_4). \]

The constants \( c, d_1, d_2, d_3 \) can be expressed in terms of the zeros of \( F(f) \),

\[
\begin{align*}
  c &= -\frac{f_1 + f_2 + f_3 + f_4}{4}, \\
  d_1 &= \frac{(f_1 + f_2 + f_3 + f_4)^2}{16} - \frac{f_1f_2 + f_2f_3 + f_3f_4 + f_1f_4 + f_1f_3 + f_3f_4}{4}, \\
  d_2 &= \frac{f_1f_2f_3 + f_1f_2f_4 + f_2f_3f_4 + f_1f_3f_4}{8}, \\
  d_3 &= -\frac{f_1f_2f_3f_4}{8}.
\end{align*}
\]

In Sec. III, we shall find the solitary wave solutions mentioned in the above proposition which correspond to special cases of the zeros \( f_1, f_2, f_3, f_4 \).

### III. EXACT SOLITARY WAVE SOLUTIONS

#### A. One double zero and two simple zeros

Let \( f_1 \) and \( f_3 \) be two different simple zeros and \( f_2 \) be a double zero of \( F(f) \). Thus we have

\[
(f')^2 = F(f) = -(f - f_1)(f - f_2)(f - f_3).
\]

Let \( f = f_2 = u \) and so \( f - f_1 = u - u_1 \), where \( u_1 = f_1 - f_2 \) and \( f - f_2 = u = u_3 \), where \( u_3 = f_3 - f_2 \). Hence the above equation becomes

\[
(u')^2 = -u^2(u - u_1)(u - u_3).
\]

Using the substitution \( u = 1/y \),

\[
(y')^2 = -y^2\left(\frac{1}{y} - u_1\right)\left(\frac{1}{y} - u_3\right) = -(1 - uy_1)(1 - uy_3) = -u_1u_3\left(\frac{1}{u_1} - y\right)\left(\frac{1}{u_3} - y\right).
\]
After some arrangements we have
\[
(y')^2 = -u_1 u_3 \left[ \left( y - \frac{1}{2} \left( \frac{1}{u_1} + \frac{1}{u_3} \right) \right)^2 - \frac{1}{4} \left( \frac{1}{u_1} - \frac{1}{u_3} \right)^2 \right].
\] (16)

Using the trigonometric substitution
\[
y - \frac{1}{2} \left( \frac{1}{u_1} + \frac{1}{u_3} \right) = \frac{1}{2} \left( \frac{1}{u_1} - \frac{1}{u_3} \right) \cosh \theta,
\]
Eq. (16) becomes
\[
(\theta')^2 = -u_1 u_3.
\]

Note that in the case when \( F(f) \) has two different simple zeros and one double zero, the solitary wave solution occurs only when we have \( f_1 < f_2 < f_3 \) and this makes \( u_1 u_3 < 0 \) or \(-u_1 u_3 > 0\). So from the above equation, we get \( \theta' = \pm \sqrt{-u_1 u_3} \) which yields
\[
\theta = \pm \sqrt{-u_1 u_3} (\xi - \xi_0),
\]
where \( \xi_0 \) is an integration constant. Hence the solution \( f \) is
\[
f = f_2 + \frac{2}{c_1 + c_2 \cosh(\sqrt{(f_2 - f_1)(f_3 - f_2)}(\xi - \xi_0))}.
\] (17)
where \( c_1 = \left( \frac{1}{f_1 - f_2} + \frac{1}{f_3 - f_2} \right) \) and \( c_2 = \left( \frac{1}{f_1 - f_2} - \frac{1}{f_3 - f_2} \right) \). It is clear that \( f \to f_2 \) as \( \xi \to \pm \infty \).

Note that when \( u_1 u_3 > 0 \) which means \( f_1 < f_3 < f_2 \) or \( f_2 < f_1 < f_3 \) we have the following solution which is not a solitary wave solution:
\[
f = f_2 + \frac{2}{c_1 \pm c_2 \sin(\sqrt{(f_2 - f_1)(f_3 - f_2)}(\xi - \xi_0))},
\] (18)
with the same \( c_1 \) and \( c_2 \) stated above.

**B. One triple zero and one simple zero**

Let \( f_1 \) be simple and \( f_2 \) be triple zeros of \( F(f) \). Hence
\[
(f')^2 = F(f) = -(f - f_1)^3(f - f_2).
\] (19)

The relations between the zeros of \( F(f) \) and the parameters are
\[
c = -\frac{f_2 + 3 f_1}{4}, \quad d_1 = \frac{f_2^3 - 6 f_1 f_2 - 3 f_1^2}{16}, \quad d_2 = \frac{3 f_1^2 f_2 + f_1^3}{8}, \quad d_3 = -\frac{f_1^3 f_2}{8}.
\] (20)

Let us solve Eq. (19). Let \( f - f_1 = u \) so (19) becomes
\[
(u')^2 = -u^3(u - u_0), \quad u_0 = f_2 - f_1.
\]

We have
\[
\frac{du}{u^{3/2} \sqrt{u_0 - u}} = \frac{\sqrt{u_0 - u} \, du}{(u_0 - u) \sqrt{u}} = d\xi.
\]

By making the substitution \( t = \sqrt{u_0 - u}/\sqrt{u} \), the above equality can be solved as
\[
-\frac{2}{u_0} \sqrt{\frac{u_0 - u}{u}} = \xi - \xi_0,
\]
where \( \xi_0 \) is an integration constant. Hence we find
\[
\frac{u_0}{u} = \frac{u_0}{1 + \frac{u_0^2}{4} (\xi - \xi_0)^2}.
\]
and inserting \( u = f - f_1 \) and \( u_0 = f_2 - f_1 \) we get the solution

\[
f = f_1 + \frac{f_2 - f_1}{1 + \frac{1}{2}(f_2 - f_1)^2(\xi - \xi_0)^2}.
\]

It is clear that \( f \to f_1 \) as \( \xi \to \pm \infty \).

\[ \text{C. Limiting cases} \]

Here we will analyze the solution (17) which corresponds to the case when \( F(f) \) has one double zero \( f_2 \) and two different simple zeros \( f_1 \) and \( f_3 \).

(a) When \( f_1 + f_3 = 2f_2 \), the solution (17) reduces to

\[
f = f_2 + \frac{2(f_2 - f_1)(f_3 - f_2)}{(f_3 - f_1)} \frac{\text{sech}(\sqrt{(f_2 - f_1)(f_3 - f_2)(\xi - \xi_0)})}{c_2 + c_1 \text{sech}(\sqrt{(f_2 - f_1)(f_3 - f_2)(\xi - \xi_0)})}.
\]

(b) When \( 2f_1f_3 = f_2(f_1 + f_3) \), the solution (17) reduces to

\[
f = \frac{c_2 \cosh(\sqrt{(f_2 - f_1)(f_3 - f_2)(\xi - \xi_0)})}{c_1 + c_2 \cosh(\sqrt{(f_2 - f_1)(f_3 - f_2)(\xi - \xi_0)})},
\]

which can be converted to

\[
f = \frac{c_2}{c_2 + c_1 \text{sech}(\sqrt{(f_2 - f_1)(f_3 - f_2)(\xi - \xi_0)})},
\]

where \( c_1 = \left(\frac{1}{f_1 - f_2} + \frac{1}{f_3 - f_2}\right) \) and \( c_2 = \left(\frac{1}{f_1 - f_2} - \frac{1}{f_3 - f_2}\right) \).

(c) When \( f_2 = 0 \), then the solution (17) reduces to

\[
f = \frac{2f_1f_3}{(f_1 + f_3) + (f_3 - f_1) \cosh(\sqrt{-f_1f_3(\xi - \xi_0)})},
\]

which can also be written as

\[
f = \frac{2f_1f_3 \text{sech}(\sqrt{-f_1f_3(\xi - \xi_0)})}{(f_3 - f_1) + (f_1 + f_3) \text{sech}(\sqrt{-f_1f_3(\xi - \xi_0)})}, \quad f_1f_3 < 0.
\]

(d) When \( f_2 \to f_1 \) or \( f_2 \to f_3 \), then the case turns to the case when \( F(f) \) has one triple zero and one simple zero. If \( f_2 \to f_1 \), the solution (17) reduces to \( f = f_1 \), and if \( f_2 \to f_3 \), the solution (17) reduces to

\[
f = f_3 + \frac{f_1 - f_3}{1 + \frac{1}{2}(f_1 - f_3)^2(\xi - \xi_0)^2}.
\]

\[ \text{IV. EXACT SOLUTIONS IN TERMS OF ELLIPTIC FUNCTIONS} \]

In this section, we will find exact solutions of (3) by using the Jacobi elliptic functions.\(^\text{12}\) Let us give the list of the Jacobi elliptic functions and first order differential equations satisfied by them.

\[ \text{A. Jacobi elliptic functions} \]

\[
y = \text{sn}u \quad (y')^2 = (1 - y^2)(1 - k^2y^2),
\]

\[
y = \text{cn}u \quad (y')^2 = (1 - y^2)(1 - k^2 + k^2y^2),
\]

\[
y = \text{dn}u \quad (y')^2 = (1 - y^2)(y^2 - 1 + k^2),
\]
B. Special solutions of (3) in terms of elliptic functions

For some special values of $c, d_1, d_2, d_3$, we have solutions of (3) in terms of Jacobi elliptic functions. Here we will present two such types of solutions.

Case 1. Solutions of the form $u(x, t) = f(\xi) = \gamma + ay(\beta \xi)$.

Here we shall find the solutions of (3) having the form $u(x, t) = f(\xi) = \gamma + ay(\beta \xi)$, where $\gamma, \alpha, \beta$ are constants, $\xi = x - ct$ and $y$ is one of the Jacobi elliptic functions. When we use this form in (5) we get the following equation:

\[
(y')^2 = -\frac{\alpha^2}{\beta^2}y^4 - \frac{4\alpha}{\beta^2}(c + \gamma)y^3 + \frac{2}{\beta^2}(2d_1 - 6c\gamma - 3\gamma^2 - 2c^2)y^2
\]

\[
+ \frac{4}{\alpha\beta^2}(2d_2 + 2d_1\gamma - 2c^2\gamma - 3c\gamma^2 - \gamma^3)y
\]

\[
+ \frac{1}{\alpha^2\beta^2}(-y^4 - 4c^2y^3 + 8d_2\gamma - 4c\gamma^3 + 8d_3 + 4d_1\gamma^2).
\]

Since the parameters are real, we have $\alpha^2/\beta^2 > 0$. Hence the coefficient of the term $y^4$ is negative. Thus there are two possibilities: $\alpha^2 = k^2\beta^2$ which corresponds to Jacobi elliptic function $cnv$ and

\[
y = \text{sn}^2v \quad (y')^2 = 4y(1 - y)(1 - k^2y),
\]

\[
y = \text{cn}^2v \quad (y')^2 = 4y(1 - y)(1 - k^2 + k^2y),
\]

\[
y = \text{dn}^2v \quad (y')^2 = 4y(1 - y)(y - 1 + k^2),
\]

\[
y = \text{tn}^2v \quad (y')^2 = 4y(1 + y)[1 + (1 - k^2)y],
\]

\[
y = \frac{1}{\text{cn}^2v} \quad (y')^2 = 4y(y - 1)[(1 - k^2)y + k^2],
\]

\[
y = \frac{1}{\text{sn}^2v} \quad (y')^2 = 4y(y - 1)[y - k^2],
\]

\[
y = \text{dn}^2v\text{tn}^2v \quad (y')^2 = 4y[(1 + y)^2 - 4k^2y].
\]
\( \alpha^2 = \beta^2 \) corresponding to \( dv \). Comparing the differential equations for \( cnv \) and \( dv \) with (42), we note that the coefficients of the terms \( y^3 \) and \( y \) should be zero. That gives

\[
\gamma = -c = \frac{f_1 + f_2 + f_3 + f_4}{4}
\]

\[
d_2 = cd_1 = \frac{f_1 + f_2 + f_3 + f_4}{64} \left[ 4(f_1 f_2 + f_1 f_3 + f_1 f_4 + f_2 f_3 + f_2 f_4 + f_3 f_4) \\
- (f_1 + f_2 + f_3 + f_4)^2 \right],
\]

where \( d_1 \) is given in (15). Note that the equality \( d_2 = cd_1 \) yields a relation between the zeros of \( F(f) \),

\[
(f_1 + f_2 - f_3 - f_4)(f_1 + f_3 - f_2 - f_4)(f_1 + f_4 - f_2 - f_3) = 0. \tag{43}
\]

Equation (42) is simplified as

\[
(y')^2 = -\frac{\alpha^2}{\beta^2}y^4 + \frac{\mu_2}{\beta^2}y^2 + \frac{\mu_0}{\alpha^2 \beta^2}, \tag{44}
\]

where

\[
\mu_2 = \frac{3}{8}(f_1 + f_2 + f_3 + f_4)^2 - (f_1 f_2 + f_1 f_3 + f_1 f_4 + f_2 f_3 + f_2 f_4 + f_3 f_4)
\]

\[
\mu_0 = \frac{(f_1 + f_2 + f_3 + f_4)^2}{16}(f_1 f_2 + f_1 f_3 + f_1 f_4 + f_2 f_3 + f_2 f_4 + f_3 f_4)
\]

\[
- \frac{5}{256}(f_1 + f_2 + f_3 + f_4)^4 - f_1 f_2 f_3 f_4.
\]

Here we shall take \( f_4 = f_1 + f_3 - f_2 \) since Eq. (43) should be satisfied. Note that we cannot have \( f_1 = f_2 \) which gives \( f_3 = f_4 \) implying two double zeros that in this case we do not have real solution \( f \). This is also same in the case of \( f_2 = f_3 \). Hence in the below computations \( f_1 \neq f_2 \) and \( f_2 \neq f_3 \).

(a) \( cn \) solution

Let \( y = cn(\beta \xi) \) with \( \xi = x - ct \) where the function \( y \) satisfies the first order differential equation (28). Hence when we compare the coefficients of (42) and (28), we get

\[
\beta^2 = \frac{\mu_2}{2k^2 - 1}, \quad \alpha^2 = \frac{k^2 \mu_2}{2k^2 - 1}, \quad k^2 = \frac{1}{2} + \frac{\mu_2}{2\sqrt{4\mu_0 + \mu_2^2}}.
\]

Explicitly, we have

\[
\beta^2 = (f_2 - f_3)(f_1 - f_2), \quad \alpha^2 = (f_1 - f_3)^2/4, \quad k^2 = (f_1 - f_3)^2/4(f_2 - f_3)(f_1 - f_2) \tag{45}
\]

or

\[
\beta^2 = (f_3 - f_2)(f_1 - f_2), \quad \alpha^2 = (f_1 + f_3 - 2f_2)^2/4, \quad k^2 = (f_1 + f_3 - 2f_2)^2/4(f_3 - f_2)(f_1 - f_2). \tag{46}
\]

Hence the corresponding solution is

\[
u(x, t) = \pm \frac{(f_1 - f_3)}{2} \operatorname{cn} \left[ \sqrt{(f_2 - f_3)(f_1 - f_2)}(x + \frac{f_1 + f_3}{2} t) \right] + \frac{f_1 + f_3}{2} \tag{47}
\]

or

\[
u(x, t) = \pm \frac{(f_1 + f_3 - 2f_2)}{2} \operatorname{cn} \left[ \sqrt{(f_2 - f_3)(f_1 - f_2)}(x + \frac{f_1 + f_3}{2} t) \right] + \frac{f_1 + f_3}{2}. \tag{48}
\]

Let us check the limiting points. It is enough to consider the parameters (45) and the solution (47). We can analyze (46) and (48) similarly. For \( k = 0 \), we have \( f_1 = f_3 \) and the solution becomes
\[ u(x, t) = f_1. \] For \( k = 1 \), we get the relation
\[ 2f_2 = f_1 + f_3. \] (49)

Hence the solution is
\[ u(x, t) = \pm \frac{(f_1 - f_3)}{2} \text{sech} \left[ \frac{f_1 - f_3}{2} \left( x + \frac{f_1 + f_3}{2} t \right) \right] + \frac{f_1 + f_3}{2}. \] (50)

(b) \( \text{dn} \) solution

Let \( y = \text{dn}(\beta \xi) \) with \( \xi = x - ct \) where the function satisfies the differential equation (29). If we compare the coefficients of (42) and (29), we get
\[ \beta^2 = \alpha^2 = \frac{2\mu_0}{-\mu_2 + \sqrt{4\mu_0 + \mu_2}}, \quad k^2 = 2 + \frac{\mu_2^2}{2\mu_0} \frac{-\mu_2}{2\mu_0} \sqrt{4\mu_0 + \mu_2}. \]

Explicitly, we have
\[ \beta^2 = \alpha^2 = (f_1 - f_3)^2/4, \quad k^2 = 4(f_1 - f_2)(f_2 - f_3)/(f_1 - f_3)^2 \] (51)

or
\[ \beta^2 = \alpha^2 = (f_1 + f_3 - 2f_2)^2/4, \quad k^2 = -4(f_1 - f_2)(f_2 - f_3)/(f_1 + f_3 - 2f_2)^2. \] (52)

Hence the solution is
\[ u(x, t) = \pm \frac{(f_1 + f_3 - 2f_2)}{2} \text{dn} \left[ \frac{f_1 + f_3 - 2f_2}{2} \left( x + \frac{f_1 + f_3}{2} t \right) \right] + \frac{f_1 + f_3}{2}. \] (53)

or
\[ u(x, t) = \pm \frac{(f_1 + f_3 - 2f_2)}{2} \text{dn} \left[ \frac{f_1 + f_3 - 2f_2}{2} \left( x + \frac{f_1 + f_3}{2} t \right) \right] + \frac{f_1 + f_3}{2}. \] (54)

Let us analyze the limiting points for the parameters (51) and the solution (53). Similar analysis can be done for (52) and (54). For \( k = 0 \), we have either \( f_1 = f_2 \) or \( f_2 = f_3 \). But we noted before that we do not have real solutions for these cases. For \( k = 1 \), from (51) we get the relation \( 2f_2 = f_1 + f_3 \). Thus the corresponding solution is
\[ u(x, t) = \pm \frac{(f_1 - f_3)}{2} \text{sech} \left[ \frac{f_1 - f_3}{2} \left( x + \frac{f_1 + f_3}{2} t \right) \right] + \frac{f_1 + f_3}{2}. \] (55)

Case 2. Solutions of the form \( u(x, t) = f(\xi) = a_1/(a_2 + b_2 y(\beta \xi)) \)

Here we shall find solutions of (3) having the form \( u(x, t) = f(\xi) = a_1/(a_2 + b_2 y(\beta \xi)) \), where \( a_1, a_2, b_2, \beta \) are constants and \( \xi = x - ct \). If we use this form in Eq. (5) we get the following equation:
\[ (y')^2 = \frac{8d_3b_2^6}{\beta^2 a_1^2} y^4 + \frac{8}{\beta^2 a_1^2} (d_2 a_1 b_2 + 4d_3 a_2 b_2) y^3 \]
\[ + \frac{4}{\beta^2 a_1^2} (12d_3 a_2^2 + 4d_1 a_1 + 6d_2 a_1 a_2 - a_1^2 c^2) y^2 \]
\[ + \frac{4}{\beta^2 a_1^2 b_2} (2a_1^2 d_1 a_2 - ca_1^4 - 2a_1^2 c^2 a_2 + 6d_2 a_1 a_2 + 8d_3 a_2^3) y \]
\[ + \frac{1}{\beta^2 a_1^2 b_2} (8d_3 a_2^2 + 8d_2 a_1 a_2 + 4a_1^2 d_1 a_2 - 4a_1^3 c^2 a_2 - 4c a_1^2 a_2 - a_1^4), \] (56)

where \( a_1, b_2, \beta \neq 0 \). As we did in the previous case we shall again use Jacobi elliptic functions (27)–(31) and study the special cases for \( k = 0 \) and \( k = 1 \). The differential equations satisfied by
these elliptic functions do not have terms with \(y^3\) and \(y\). Hence the coefficients of \(y^3\) and \(y\) should be zero in (56). Let also \(a = a_2/a_1\) and \(b = b_2/a_1\), \(a_1 \neq 0\). Then we get

\[
d_1 = \frac{c}{2a} + c^2 + 8a^2 d_3 \]

\[
d_2 = -4d_3a = (f_1 f_2 f_3 + f_2 f_3 f_4 + f_3 f_4 f_4) / 8 \]

\[
a = (f_1 f_2 f_3 + f_2 f_3 f_4 + f_3 f_4 f_4) / 4 f_1 f_2 f_3 f_4 \]

with a relation between the zeros of \(F(f)\),

\[
(f_1 f_2 f_3 - f_2 f_3 f_4 - f_1 f_2 f_4 + f_1 f_3 f_4)(f_1 f_2 f_3 - f_2 f_3 f_4 + f_1 f_2 f_4 - f_1 f_3 f_4) \times (f_1 f_2 f_3 + f_2 f_3 f_4 - f_1 f_2 f_4 - f_1 f_3 f_4) = 0. \tag{57} \]

Hence (56) is simplified as

\[
(y')^2 = \frac{1}{\beta^2} v_4 y^4 + \frac{1}{\beta^2} v_2 y^2 + \frac{1}{\beta^2 b^2} v_0, \tag{58} \]

where

\[
v_4 = -b^2 f_1 f_2 f_3 f_4 \]

\[
v_2 = \frac{(f_1 f_2 f_3 + f_1 f_2 f_4 + f_1 f_3 f_4 + f_2 f_3 f_4)^2}{8 f_1 f_2 f_3 f_4} - \frac{2 f_1 f_2 f_3 f_4 (f_1 + f_2 + f_3 + f_4)}{f_1 f_2 f_3 + f_1 f_2 f_4 + f_1 f_3 f_4 + f_2 f_3 f_4} \]

\[
v_0 = \frac{(f_1 f_2 f_3 + f_1 f_2 f_4 + f_1 f_3 f_4 + f_2 f_3 f_4)^4}{8 f_1 f_2 f_3 f_4} - \frac{(f_1 f_2 f_3 + f_1 f_2 f_4 + f_1 f_3 f_4 + f_2 f_3 f_4)^4}{256 (f_1 f_2 f_3 f_4)^3} - 1, \]

for \(f_1 f_2 f_3 f_4 \neq 0\). If any one of the roots of \(F\) is zero, i.e., \(f_1 f_2 f_3 f_4 = 0\), then (57) implies that one more root is also zero. Hence in such a case \(F\) has a double zero and two simple zeros. This case was studied in Sec. III A.

Now let us study the elliptic functions satisfying (58). Note that if \(f_1 f_2 f_3 f_4 \neq 0\), we will take \(f_4 = f_1 f_2 f_3 (f_2 f_3 + f_1 f_2 - f_1 f_3)\) by the relation (57) in the below computations.

(a) \(sn\) solution

Let \(y = \text{sn}(\beta \xi)\) with \(\xi = x - ct\) where the function \(y\) satisfies the first order differential equation (27). Then when we compare the coefficients of (58) and (27), we get

\[
\beta^2 = -v_4 - v_2, \quad k^2 = -\frac{v_4}{v_4 + v_2}, \quad b^2 = -\frac{v_0}{v_4 + v_2}. \tag{59} \]

Explicitly, we have

\[
\beta^2 = \frac{2b^2 f_1^2 f_2^2 f_3^2 - f_1^2 f_2^2 + 2 f_1^2 f_2 f_3 - 2 f_2^2 f_3^2 + 2 f_2 f_3 f_1 - f_2^2 f_2^2}{f_2 f_3 + f_1 f_2 - f_1 f_3} \]

\[
k^2 = \frac{2b^2 f_1^2 f_2^2 f_3^2 - f_1^2 f_2^2 + 2 f_1^2 f_2 f_3 - 2 f_2^2 f_3^2 + 2 f_2 f_3 f_1 - f_2^2 f_2^2}{2b^2 f_1^2 f_2^2 f_3^2 - f_1^2 f_2^2 + 2 f_1^2 f_2 f_3 - 2 f_2^2 f_3^2 + 2 f_2 f_3 f_1 - f_2^2 f_2^2}. \tag{60} \]
and we obtain four choices for the value $b$: $\pm(f_1 - f_3)/2f_1f_3$, $\pm(f_1f_2 - 2f_1f_3 + f_2f_3)/2f_1f_2f_3$. Taking $b = (f_1 - f_3)/2f_1f_3$ yields

$$\beta^2 = -\frac{f_2^2(f_1 + f_3)^2 - 4f_1f_3(f_1f_2 + f_2f_3 - f_1f_3)}{4(f_1f_2 + f_2f_3 - f_1f_3)},$$

$$k^2 = \frac{f_2^2(f_1 - f_3)^2}{f_2^2(f_1 + f_3)^2 - 4f_1f_3(f_1f_2 + f_2f_3 - f_1f_3)}.$$  \hspace{1cm} (61)

Hence the solution is

$$u(x, t) = \frac{2f_1f_3}{(f_1 + f_3) + (f_1 - f_3)\sin[\beta(x - ct)]},$$

where

$$c = -\frac{f_1 + f_2 + f_3}{4} - \frac{f_1f_2f_3}{4(f_2f_3 + f_1f_2 - f_1f_3)}.$$  \hspace{1cm} (62)

Let us study the limiting cases. For $k = 0$, there are two possibilities: $f_2 = 0$ or $f_1 = f_3$. If $f_2 = 0$, then $\beta = \pm\sqrt{f_1f_3}$, $f_1f_3 > 0$ and the corresponding solution is

$$u(x, t) = \frac{2f_1f_3}{(f_1 + f_3) \pm (f_1 - f_3)\sin[\sqrt{f_1f_3}(x + \frac{f_1f_3}{4}ct)]}. \hspace{1cm} (63)$$

If $f_1 = f_3$, then $a = 1/f_1$ and $b = 0$ so we have constant solution $u(x, t) = f_1$. For $k = 1$ then from (61) we have

$$4f_1f_3(f_2 - f_1)(f_2 - f_3) = 0.$$  

It is not possible to have $f_1 = 0$ or $f_3 = 0$ because of the definition of $b$. If $f_1 = f_2$ or $f_2 = f_3$ we have $\beta^2 \leq 0$. Hence we do not have real solution for $k = 1$.

(b) cn solution

Let $v = \text{cn}(\beta \xi)$ with $\xi = x - ct$ where the function $v$ satisfies the first order differential equation (28). If we compare the coefficients of (58) and (28), we get

$$\beta^2 = -2v_4 - v_2, \quad k^2 = \frac{v_4}{2v_4 + v_2}, \quad b^2 = -\frac{v_0}{v_4 + v_2}. \hspace{1cm} (64)$$

Since we have the same relation for $b$ as in the Case 2(a), we may also take $b = (f_1 - f_3)/2f_1f_3$. Hence (64) becomes

$$\beta^2 = \frac{f_1f_3(f_1 - f_2)(f_2 - f_3)}{f_2f_3 + f_1f_2 - f_1f_3}, \quad k^2 = \frac{f_2^2(f_1 - f_3)^2}{4f_1f_3(f_1 - f_2)(f_2 - f_3)}. \hspace{1cm} (65)$$

Thus the solution is

$$u(x, t) = \frac{2f_1f_3}{(f_1 + f_3) + (f_1 - f_3)\text{cn}[\beta(x - ct)]},$$

where

$$c = -\frac{f_1 + f_2 + f_3}{4} - \frac{f_1f_2f_3}{4(f_2f_3 + f_1f_2 - f_1f_3)}.$$  \hspace{1cm} (66)

For $k = 0$, there are two possibilities: $f_2 = 0$ or $f_1 = f_3$. If $f_2 = 0$, then $\beta = \pm\sqrt{f_1f_3}$, $f_1f_3 > 0$ and the corresponding solution is

$$u(x, t) = \frac{2f_1f_3}{(f_1 + f_3) + (f_1 - f_3)\cos[\sqrt{f_1f_3}(x + \frac{f_1f_3}{4}ct)]}. \hspace{1cm} (67)$$

If $f_1 = f_3$, then $a = 1/f_1$ and $b = 0$ so we have a constant solution $u(x, t) = f_1$. For $k = 1$, we have the following relation from (65):

$$2f_1f_3 = f_2(f_1 + f_3). \hspace{1cm} (68)$$
Hence the solution is

$$u(x, t) = \frac{2f_1f_3}{(f_1 + f_3) + (f_1 - f_3)\text{sech}\left[\frac{f_1f_3 - f_3}{f_1 + f_3} \sqrt{f_1f_3(x - ct)}\right]} ,$$

(69)

where $c = -(f_1^2 + 6f_1f_3 + f_3^2)/4(f_1 + f_3)$.

(c) \text{dn} solution

Let $y = \text{dn}(\beta \xi)$ with $\xi = x - ct$ where the function $y$ satisfies the first order differential equation (29). When we compare the coefficients of (58) and (29), we get

$$\beta^2 = -v_4, \quad k^2 = \frac{2v_4 + v_2}{v_4}, \quad b^2 = -\frac{v_0}{v_4 + v_2} .$$

(70)

Same as before let us take $b = (f_1 - f_3)/2f_1f_3$. Hence (70) becomes

$$\beta^2 = \frac{f_3^2(f_1 - f_3)^2}{4(f_2f_3 + f_1f_2 - f_1f_3)}, \quad k^2 = \frac{4f_1f_3(f_1 - f_2)(f_2 - f_3)}{f_3^2(f_1 - f_3)^2} .$$

(71)

Thus the solution is

$$u(x, t) = \frac{2f_1f_3}{(f_1 + f_3) + (f_1 - f_3)\text{dn}[\beta(x - ct)]} ,$$

(72)

where

$$c = -\frac{f_1 + f_2 + f_3}{4} - \frac{f_1f_2f_3}{4(f_2f_3 + f_1f_2 - f_1f_3)} .$$

For $k = 0$, there are four possibilities: $f_1 = 0$, $f_3 = 0$, $f_1 = f_2$, or $f_2 = f_3$. We cannot have $f_1 = 0$ or $f_3 = 0$ because of the definition of $b$. If $f_1 = f_2$ or $f_2 = f_3$, the solution is $u(x, t) = f_3$. For $k = 1$, we have $2f_1f_3 = f_2(f_1 + f_3)$. So the corresponding solution is

$$u(x, t) = \frac{2f_1f_3}{(f_1 + f_3) + (f_1 - f_3)\text{sech}\left[\frac{f_1f_3 - f_3}{f_1 + f_3} \sqrt{f_1f_3(x - ct)}\right]}, \quad f_1f_3 > 0 ,$$

(73)

where $c = -(f_1^2 + 6f_1f_3 + f_3^2)/4(f_1 + f_3)$.

(d) \text{tn} solution

Let $y = \text{tn}(\beta \xi)$ with $\xi = x - ct$ where the function $y$ satisfies the first order differential equation (30). Hence when we compare the coefficients of (58) and (30), we get

$$\beta^2 = v_2 - v_4, \quad k^2 = \frac{v_2 - 2v_4}{v_2 - v_4}, \quad b^2 = \frac{v_0}{v_2 - v_4} .$$

(74)

Here we notice that third equality of (74) reveals that $b$ is not real for any values of $k$. Hence for all values of $k^2 \in [0, 1]$ we do not have real solution.

(e) $1/\text{sn}$ solution

Let $y = 1/\text{sn}(\beta \xi)$ with $\xi = x - ct$ where the function $y$ satisfies the first order differential equation (31). Hence when we compare the coefficients of (58) and (31), we get

$$\beta^2 = v_4, \quad k^2 = -\frac{v_2 - v_4}{v_4}, \quad b^2 = -\frac{v_0}{v_2 + v_4} .$$

(75)

If we take $b = (f_1 - f_3)/2f_1f_3$, (75) becomes

$$\beta^2 = -\frac{f_3^2(f_1 - f_3)^2}{4(f_2f_3 + f_1f_2 - f_1f_3)}, \quad k = \pm \frac{f_2(f_1 + f_3) - 2f_1f_3}{f_3(f_1 - f_3)} .$$

(76)

The corresponding solution is

$$u(x, t) = \frac{2f_1f_3\text{sn}[\beta(x - ct)]}{(f_1 + f_3)\text{sn}[\beta(x - ct)] + (f_1 - f_3)} ,$$

(77)

where

$$c = -\frac{f_1 + f_2 + f_3}{4} - \frac{f_1f_2f_3}{4(f_2f_3 + f_1f_2 - f_1f_3)} .$$
For $k = 0$, we have the relation $f_2(f_1 + f_3) = 2f_1f_3$ and the solution becomes

$$u(x, t) = \frac{2f_1f_3 \sin \left[ \frac{\beta - ct}{f_1 + f_3} \sqrt{-f_1f_3(x - ct)} \right]}{(f_1 + f_3) \sin \left[ \frac{\beta - ct}{f_1 + f_3} \sqrt{-f_1f_3(x - ct)} \right]} \pm (f_1 - f_3), \quad -f_1f_3 > 0, \tag{78}$$

where $c = -(f_1^2 + 6f_1f_3 + f_3^2)/4(f_1 + f_3)$. For $k = 1$, then from (76) we have

$$4f_1f_3(f_2 - f_1)(f_2 - f_3) = 0.$$

It is not possible to have $f_1 = 0$ or $f_3 = 0$ because of the definition of $b$. If $f_1 = f_2$ or $f_2 = f_3$, we have $\beta^2 < 0$. Hence we do not have real solution for $k = 1$.

(f) 1/cn solution

Let $y = 1/cn(\beta \xi)$ with $\xi = x - ct$ where the function $y$ satisfies the first order differential equation (32). If we compare the coefficients of (58) and (32), we get

$$\beta^2 = v_2 + 2v_4, \quad k^2 = \frac{v_2 + v_4}{v_2 + 2v_4}, \quad b^2 = -\frac{v_0}{v_2 + v_4}. \tag{79}$$

Since we take $b = (f_1 - f_3)/2f_1f_3$, (79) becomes

$$\beta^2 = \frac{f_1f_3(f_1 - f_3)(f_2 - f_3)}{f_2f_3 + f_1f_2 - f_1f_3}, \quad k^2 = \frac{[f_2(f_1 + f_3) - 2f_1f_3]^2}{4f_1f_3(f_1 - f_2)(f_2 - f_3)}.$$

The corresponding solution is

$$u(x, t) = \frac{2f_1f_3 \text{cn}[\beta(x - ct)]}{(f_1 + f_3) \text{cn}[\beta(x - ct)] + (f_1 - f_3)}, \tag{80}$$

where

$$c = \frac{-f_1 + f_2 + f_3}{4} - \frac{f_1f_2f_3}{4(f_2f_3 + f_1f_2 - f_1f_3)}.$$

For $k = 0$, we have the relation $f_2(f_1 + f_3) = 2f_1f_3$ and the solution becomes

$$u(x, t) = \frac{2f_1f_3 \cos \left[ \frac{\beta - ct}{f_1 + f_3} \sqrt{-f_1f_3(x - ct)} \right]}{(f_1 + f_3) \cos \left[ \frac{\beta - ct}{f_1 + f_3} \sqrt{-f_1f_3(x - ct)} \right]} + (f_1 - f_3), \quad -f_1f_3 > 0, \tag{81}$$

where $c = -(f_1^2 + 6f_1f_3 + f_3^2)/4(f_1 + f_3)$. The case for $k = 1$ gives the condition $f_2^2(f_1 - f_3)^2 = 0$ to be satisfied. Hence we have two possibilities: $f_2 = 0$ or $f_1 = f_3$. If $f_2 = 0$, then $\beta = \pm \sqrt{-f_1f_3}$, $-f_1f_3 > 0$ and the corresponding solution is

$$u(x, t) = \frac{2f_1f_3 \text{sech}[\sqrt{-f_1f_3(x + \frac{\beta + f_3}{4})}]}{(f_1 + f_3) \text{sech}[\sqrt{-f_1f_3(x + \frac{\beta + f_3}{4})}] + (f_1 - f_3)}, \quad -f_1f_3 > 0. \tag{82}$$

If $f_1 = f_3$, then $\alpha = 1/f_1$ and $b = 0$ so we have a constant solution $u(x, t) = f_1$.

(g) dn solution

Let $y = \text{dn}(\beta \xi) \text{tn}(\beta \xi)$ with $\xi = x - ct$ where the function $y$ satisfies the first order differential equation (33). Hence when we compare the coefficients of (58) and (33), we get

$$\beta^2 = v_4, \quad k^2 = \frac{2v_4 - v_2}{4v_4}, \quad b^2 = \frac{v_0}{v_4}.$$

The third equality above gives four choices for $b$,

$$\pm \frac{\sqrt{-f_2(f_3 - f_1)(2f_1f_3 - f_2)(f_1 + f_3))}}{f_1f_2f_3} \quad \pm \frac{\sqrt{-f_2(f_3 - f_1)(2f_1f_3 - f_2)(f_1 + f_3))}}{f_1f_2f_3}. \tag{83}$$

To have real solutions, the parameters must be real. Hence from the expressions for $b$ we have either $-f_2(f_3 - f_1)(2f_1f_3 - f_2(f_1 + f_3)) \geq 0$ or $f_2(f_3 - f_1)(2f_1f_3 - f_2(f_1 + f_3)) \geq 0$. If the first one
This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 176.239.11.64 On: Tue, 02 Sep 2014 11:08:49

\[
\beta^2 = \frac{f_2(f_3 - f_1)(2f_1f_3 - f_2(f_1 + f_3))}{4f_2f_3 + f_1f_2 - f_1f_3}, \quad k^2 = \frac{-f_1^2(f_1 - f_2)^2}{f_2(f_3 - f_1)(2f_1f_3 - f_2(f_1 + f_3))}.
\]

If the second one is true then
\[
\beta^2 = \frac{f_2(f_1 - f_3)(2f_1f_3 - f_2(f_1 + f_3))}{4f_2f_3 + f_1f_2 - f_1f_3}, \quad k^2 = \frac{f_1^2(f_2 - f_3)^2}{f_2(f_1 - f_3)(-2f_1f_3 + f_2(f_1 + f_3))}.
\]

From the equality for \( k^2 \) in (84) we get
\[
\frac{k^2 - 1}{k^2} = \frac{f_1^2(f_3 - f_2)^2}{f_2(f_2 - f_1)^2} \geq 0.
\]

This gives that \( k^2 \geq 1 \). We know that for the parameter \( k^2 \) of Jacobi elliptic functions we have \( 0 \leq k^2 \leq 1 \). Additionally, at the limiting points \( k = 0 \) and \( k = 1 \) it yields that \( F(f) \) has two double zeros that is the case which does not give real solution as we stated in Sec. II B. We also have the similar result for (85). Hence we do not have real solutions for all \( k^2 \in [0, 1] \).

C. Discussion about the special solutions

When \( F(f) \) has one double \( f_2 \) and two simple zeros \( f_1 \) and \( f_3 \) we have the following system of equations:
\[
\begin{align*}
-4c &= f_1 + 2f_2 + f_3 \\
4(d_1 - c^2) &= -[f_1f_3 + 2f_2(f_1 + f_3) + f_2^2] \\
8d_2 &= 2f_1f_2f_3 + f_2^2(f_1 + f_3) \\
8d_3 &= -f_1f_2^2f_3.
\end{align*}
\]

The exact solutions in terms of the Jacobi elliptic functions take the following forms:

(i) In Cases 1(a) and 1(b), we have \( d_2 = cd_1 \). Using this in (86) we obtain that either \( 2f_2 = f_1 + f_3 \) or \( f_1 = f_3 \). The second one is not allowed due to the discussion in Sec. III B. By using (86), the first one leads to \( k^2 = 1 \). In this case, the solution is given in (50) and (55) which are compatible with the limiting solutions discussed in Sec. V, part (a).

(ii) In Cases 2(b) and 2(c), we have \( d_2 = -4d_1a \). From the first equation of (86) we have \( a = 1/f_2 \). Then this implies \( d_2 = -4d_1/f_2 \). This constraint gives \( 2f_1f_3 = f_2(f_1 + f_3) \) which yields \( k^2 = 1 \). In this case, the solutions are given in (69) and (73) which are compatible with the limiting solutions discussed in Sec. V, part (b).

(iii) If \( f_2 = 0 \), then \( d_3 = 0 \) hence \( d_2 = 0 \) which leads to \( k^2 = 1 \). In this case, the solution is (82) given in Case 2(t) which are compatible with the limiting solutions discussed in Sec. V, part (c).

D. General solutions of (3) in terms of elliptic functions

Here we shall deal with the most general form of solutions
\[
u(x, t) = f(\xi) = (a_1 + b_1y(\beta \xi))/(a_2 + b_2y(\beta \xi), \quad \xi = x - ct.
\]

When we insert this form into the equation \( F(f) \) we get
\[
(y')^2 = \frac{1}{\beta^2(b_1a_2 - b_2a_1)^2} (\Omega_4 y^4 + \Omega_2 y^2 + \Omega_1 y + 0),
\]

where
\[
\begin{align*}
\Omega_4 &= -b_1^2(b_1 + 2cb_2)^2 + 4b_2^2(b_1^2d_1 + 2b_1b_2d_2 + 2b_2^2d_3) = b_1^2 F(b), \quad b = b_1/b_2, \\
\Omega_3 &= 4(2d_2a_1b_2^2 - cb_1^2a_2 + 8d_3a_2b_2^3 - 3ca_1b_2^3b_2 + 2d_1a_1b_2^2 + 2d_1b_1^2a_2b_2 \\
&- 2c^2a_1b_2^2 - 2c^2b_1^2a_2b_2 + 6d_2b_1a_2b_2^2 - a_1b_1^4),
\end{align*}
\]
In (88), we have effectively four independent parameters. By choosing these constants properly we get several solutions in terms of elliptic functions. We can analyze these solutions in two groups:

(i) If \(F(b)\) has zeros, then we can make the coefficients of \(y^4\) to vanish by taking \(F(b) = 0\).

This means that \(b = b_1/b_2\) is a zero of \(F(b)\). In addition to that choosing the constant \(\Omega_0 = 0\) yields that \(F(a) = 0\) where \(a = a_1/a_2\). This also means that \(a = a_1/a_2\) is another zero of \(F(b)\). Note that \(a \neq b\) since \(b_1 a_2 - b_2 a_1 \neq 0\). Then Eq. (88) takes the form where the square of elliptic functions and their inverses given in (34)–(40) satisfy. By making substitution \(a = a_1/a_2\) and \(b = b_1/b_2\), Eq. (88) becomes

\[
(y)^2 = \frac{b_2^2 F(b)}{\beta^2 a_2^2 (b-a)^3} y^4 + \frac{4b_2}{\beta^2 a_2^2 (b-a)^2} \omega_3 y^3 + \frac{2}{\beta^2 (b-a)^2} \omega_2 y^2
\]

where

\[
\omega_3 = 2ad_3 + c b^3 + 8d_3 - 3acb^2 + 2d_1 ab + 2d_1 b^2 - 2c^2 ab - 2c^2 b^2 + 6d_2 b - ab^3
\]

\[
\omega_2 = 2d_1 a^2 + 2d_1 b^2 - 2c^2 a^2 - 2c^2 b^2 + 24d_3 - 6c a^2 b - 6c a b^2 + 12d_2 a + 12d_2 b
\]

\[-3a^2 b^2 + 8d_1 ab - 8c^2 ab
\]

\[
\omega_1 = 2d_2 b - ca^3 + 8d_3 - 3cba^2 + 2d_1 ab + 2d_1 a^2 - 2c^2 ab - 2c^2 a^2 + 6d_2 a - ba^3.
\]

If \(a\) and \(b\) are the zeros of \(F(f)\), then \(F(a) = F(b) = 0\) and we do not have the terms with \(y^4\) and the constant term in (94). For instance, let \(a = f_1\) and \(b = f_2\), then we can write \(F(f) = -(f - f_1)(f - f_2)(f - f_3)(f - f_4)\) such that \(f_1, f_2, f_3, f_4\) are zeros of \(F(f)\). Let us write \(\omega_1, \omega_2, \omega_3\) in terms of the zeros of the function \(F(f)\) by the help of (15),

\[
\omega_1 = \frac{(f_1 - f_2)^2}{4} (f_1 - f_4)(f_1 - f_3),
\]

\[
\omega_2 = \frac{1}{2} (f_1 - f_2)^2 ((f_2 - f_3)(f_1 - f_4) + (f_1 - f_3)(f_2 - f_4)),
\]

\[
\omega_3 = \frac{(f_1 - f_2)^2}{4} (f_2 - f_3)(f_2 - f_4).
\]

Now we give all solutions of (3) of the form (87). Let \(y = sn^2(\beta \xi)\) with \(\xi = x - ct\) where the function \(y\) satisfies the first order differential equation (34). Hence when we compare the coefficients of (94)(34), we get

\[
\beta^2 = \frac{\omega_1 \omega_3}{k^2 (b-a)^2}, \quad \frac{b_2}{a_2} = \frac{-2(1+k^2)\omega_1}{\omega_2}
\]

\[
k^2 = 1 + \frac{\omega_2^2}{8\omega_1 \omega_3} \pm \frac{\omega_2}{8\omega_1 \omega_3} \sqrt{\omega_2^2 - 16\omega_1 \omega_3}.
\]
Here without losing any generality we take \( f_1 \leq f_2 \leq f_3 \leq f_4 \). In the case of equality between the zeros of \( F(f) \) we have either one double zero and two simple zeros or one triple zero and one simple zero cases to have real solutions. Both of these cases were studied in Secs. IIIA and IIIB. Therefore we assume, in the sequel, that we have \( f_1 < f_2 < f_3 < f_4 \).

1. Let \( a = f_1, b = f_2 \). For this choice we have

\[
\beta = \frac{1}{2} \sqrt{(f_1 - f_3)(f_2 - f_4)}, \quad \frac{b_2}{a_2} = \frac{f_2 - f_1}{f_2 - f_4}, \quad k^2 = \frac{(f_2 - f_3)(f_1 - f_4)}{(f_1 - f_3)(f_2 - f_4)},
\]

and hence the solution with the initial condition \( f(0) = f_1 \) is

\[
u(x, t) = f(\xi) = \frac{f_1(f_2 - f_4) + f_2(f_4 - f_1)sn^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)\xi})}{(f_2 - f_4) + (f_4 - f_1)sn^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)\xi})}.
\]

(98)

Note that since \( a \) and \( b \) are any zeros of \( F(f) \) we have other choices of these parameters.

2. Let \( a = f_2, b = f_1 \). For this choice we have

\[
\beta = \frac{1}{2} \sqrt{(f_1 - f_3)(f_2 - f_4)}, \quad \frac{b_2}{a_2} = \frac{f_3 - f_2}{f_1 - f_3}, \quad k^2 = \frac{(f_2 - f_4)(f_1 - f_3)}{(f_1 - f_4)(f_2 - f_3)},
\]

and hence the solution with the initial condition \( f(0) = f_2 \) is

\[
u(x, t) = f(\xi) = \frac{f_2(f_1 - f_3) + f_1(f_3 - f_2)sn^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)\xi})}{(f_1 - f_3) + (f_3 - f_2)sn^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)\xi})}.
\]

(99)

3. Let \( a = f_3, b = f_4 \). For this choice we have

\[
\beta = \frac{1}{2} \sqrt{(f_1 - f_3)(f_2 - f_4)}, \quad \frac{b_2}{a_2} = \frac{f_2 - f_3}{f_4 - f_2}, \quad k^2 = \frac{(f_4 - f_3)(f_1 - f_4)}{(f_1 - f_3)(f_4 - f_2)},
\]

and hence the solution with the initial condition \( f(0) = f_3 \) is

\[
u(x, t) = f(\xi) = \frac{f_3(f_4 - f_2) + f_4(f_2 - f_3)sn^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)\xi})}{(f_4 - f_2) + (f_2 - f_3)sn^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)\xi})}.
\]

(100)

4. Let \( a = f_4, b = f_3 \). For this choice we have

\[
\beta = \frac{1}{2} \sqrt{(f_1 - f_3)(f_2 - f_4)}, \quad \frac{b_2}{a_2} = \frac{f_1 - f_4}{f_3 - f_1}, \quad k^2 = \frac{(f_3 - f_2)(f_4 - f_1)}{(f_1 - f_3)(f_4 - f_2)},
\]

and hence the solution with the initial condition \( f(0) = f_4 \) is

\[
u(x, t) = f(\xi) = \frac{f_4(f_3 - f_1) + f_3(f_1 - f_4)sn^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)\xi})}{(f_3 - f_1) + (f_1 - f_4)sn^2((1/2)\sqrt{(f_1 - f_3)(f_2 - f_4)\xi})}.
\]

(101)

Similarly, we can also find other type of solutions including square of Jacobi elliptic functions and inverses of them. But they are equivalent because of the relations \( cn^2 v + sn^2 v = 1 \) and \( tn^2 v = sn^2 v / (1 - sn^2 v) \). Hence the solutions given in (1), (2), (3), and (4) are all the most general solutions of (3) depending upon the initial conditions.

(ii) Another choice is taking \( a_1, b_1, a_2, b_2 \) so that \( \Omega_1 = \Omega_3 = 0 \). Then Eq. (88) takes the form where elliptic functions and their inverses given in (27)–(32) satisfy. Note that if we take \( b_1 = 0 \) to make \( \Omega_1 = \Omega_3 = 0 \) we have \( c = -a_1/a_2, d_2 = cd_1 \), and the solution becomes \( f(\xi) = \gamma + \alpha y \) which we have already studied in Sec. IVB, Case 1. If \( a_2 = 0 \) we can use inverse of Jacobi elliptic functions for \( y \) and, then the case turns to \( b_2 = 0 \) case. When we take \( b_1 = 0 \), to make \( \Omega_1 = \Omega_2 = 0 \) we have \( d_2 = -4d_1 a_2/a_1 \) and \( d_1 = 8d_1 a_2/a_1^2 + c^2 + c/2a_2 \) and the solution becomes \( f(\xi) = 1/(\alpha + \gamma y) \) that is the case we have already studied in Sec. IVB, Case 2.

In Sec. V, we mention about the system (2) when \( \ell = 3 \) and \( \ell = 4 \).
V. $\ell = 3$ AND $\ell = 4$ CASES

(1) The degenerate coupled KdV equation for $\ell = 3$ is

\[ u_t = \frac{3}{2} u u_x + v_x \]
\[ v_t = v u_x + \frac{1}{2} u v_x + \omega_x \]
\[ \omega_t = -\frac{1}{4} u_{xxx} + \omega u_x + \frac{1}{2} u \omega_x. \] (102)

Here we will show that unlike the case $\ell = 2$, we have real traveling wave solution with asymptotically vanishing boundary condition in $\ell = 3$ case. Let $u(x, t) = f(\xi)$, $v(x, t) = g(\xi)$, and $\omega(x, t) = h(\xi)$, where $\xi = x - ct$. From the first equation of (102) we have

\[ -c f'' = \frac{3}{2} f f' + g', \]

which gives

\[ g(\xi) = -c f - \frac{3}{4} f^2 + d_1, \]

where $d_1$ is an integration constant. Using $g(\xi)$ in the second equation of (102) yields

\[ h' = 3 c f f' + \frac{3}{2} f^3 f' + (c^2 - d_1) f'. \]

Integrating above equation once we get

\[ h(\xi) = \frac{3}{2} c f^2 + \frac{1}{2} f^3 + (c^2 - d_1) f + d_2, \]

where $d_2$ is an integration constant. Using $h(\xi)$ in the third equation of (102) yields

\[ \frac{1}{4} f''' = \left( \frac{9 c^2}{4} - \frac{3 d_1}{2} \right) f f' + \frac{9 c^2}{2} f^2 f' + \frac{5}{4} f^3 f' + (c^3 - c d_1 + d_2) f'. \]

Integrating above equation once we obtain

\[ \frac{1}{4} f'' = \left( \frac{9 c^2}{4} - \frac{3 d_1}{2} \right) f^2 + \frac{3 c^2}{2} f^3 + \frac{5}{16} f^4 + (c^3 - c d_1 + d_2) f + d_3. \]

By using $f'$ as an integrating factor, we integrate once more. Finally, we get

\[ \left( f' \right)^2 = \frac{f^5}{2} + 3 c f^4 + (6 c^2 - 2 d_1) f^3 + 4(c^3 - c d_1 + d_2) f^2 + 8 d_3 f + 8 d_4, \]

where $c, d_1, d_2, d_3, d_4$ are constants. If we apply the boundary conditions $f, f', f'', f'''$, $g, g', h, h'$ to $\pm \infty$ we get $d_1 = d_2 = d_3 = d_4 = 0$. Hence we have

\[ \left( f' \right)^2 = \frac{f^5}{2} + 3 c f^4 + 6 c^2 f^3 + 4 c^3 f^2 \]
\[ = \frac{f^2}{2} (f + 2c)^3. \] (103)

By using trigonometric substitution $f = -2c \sin^2 \theta$ and making the cancelations, above equality becomes

\[ \frac{d\theta}{c^{3/2} \sin \theta \cos^2 \theta} = \mp d\xi \quad \Rightarrow \quad \frac{\sin \theta d\theta}{c^{3/2} \sin^2 \theta \cos^2 \theta} = \mp d\xi. \]

Making the substitution $u = \cos \theta$ gives

\[ \frac{du}{c^{3/2} (u^2 - 1) u^2} = \mp d\xi, \]

This article is copyrighted as indicated in the article. Reuse of AIP content is subject to the terms at: http://scitation.aip.org/termsconditions. Downloaded to IP: 176.239.11.64 On: Tue, 02 Sep 2014 11:08:49
which is solved as
\[
\frac{1}{c^{3/2}} \left\{ \frac{1}{u} + \frac{1}{2} \ln |u - 1| - \frac{1}{2} \ln |u + 1| \right\} = \mp (\xi - \xi_0), \tag{104}
\]
where \(\xi_0\) is an integration constant. Note that \(u = \cos \theta = \pm \left(1 + \frac{f}{2c}\right)^{1/2}\). When the solution \(f = 0\), \(u\) is either 1 or -1. Insert the expression for \(u\) into the above equation so we get the relation defining the solution \(f\),
\[
\frac{1}{c^{3/2}} \left\{ \pm \left(1 + \frac{f}{2c}\right)^{-1/2} + \ln \left| \pm \left(1 + \frac{f}{2c}\right)^{1/2} - 1 \right| \right\} = \mp (\xi - \xi_0). \tag{105}
\]
Hence we have asymptotically vanishing real traveling solution for \(\ell = 3\). We expect that this is true for all odd \(\ell\).

(2) Now let us analyze \(\ell = 4\) case. The degenerate coupled KdV equation for \(\ell = 4\) is
\[
\begin{align*}
u_t &= \frac{3}{2} uu_x + v_x \\
v_t &= vu_x + \frac{1}{2} u v_x + \omega_x \\
\omega_t &= \omega u_x + \frac{1}{2} u \omega_x + \rho_x \\
\rho_t &= -\frac{1}{4} u_{xxx} + \rho u_x + \frac{1}{2} u \rho_x. \tag{106}
\end{align*}
\]

**Proposition 5.1.** There is no real asymptotically vanishing traveling wave solution of Eq. (106) in the form \(u(x, t) = f(\xi), v(x, t) = g(\xi), \omega(x, t) = h(\xi), \) and \(\rho(x, t) = r(\xi)\), where \(\xi = x - ct\).

**Proof.** Let \(u(x, t) = f(\xi), v(x, t) = g(\xi), \omega(x, t) = h(\xi), \) and \(\rho(x, t) = r(\xi)\), where \(\xi = x - ct\). From the first equation of (106) we have
\[-cf' = \frac{3}{2} f f' + g',
\]
which gives
\[g(\xi) = -cf - \frac{3}{4} f^2 + d_1,
\]
where \(d_1\) is an integration constant. Using \(g(\xi)\) in the second equation of (106) yields
\[h' = 3cf f' + \frac{3}{2} f^2 f' + (c^2 - d_1) f'.
\]
Integrating above equation once we have
\[h(\xi) = \frac{3}{2} cf^2 + \frac{1}{2} f^3 + (c^2 - d_1) f + d_2,
\]
where \(d_2\) is an integration constant. Using \(h(\xi)\) in the third equation of (106) yields
\[r' = -\frac{5}{4} f^3 f' - \frac{9}{2} cf^2 f' - \left( \frac{9}{2} c^2 + \frac{3}{2} d_1 \right) f f' + (-c^3 + cd_1 - d_2) f'.
\]
Integrating this equation once gives
\[r(\xi) = -\frac{5}{16} f^4 - \frac{3}{2} cf^3 + \left( -\frac{9}{4} c^2 + \frac{3}{4} d_1 \right) f^2 + (-c^3 + cd_1 - d_2) f + d_3,
\]
where $d_3$ is an integration constant. Using $r(\xi)$ in the fourth equation of (106) gives

$$\frac{1}{4} f'''' = -\frac{15}{16} f^4 f' - 5 c f^3 f' + \left(\frac{3}{2} d_1 - 9 c^2\right) f^2 f' + \left(3 c d_1 - 6 c^3 - \frac{3}{2} d_2\right) f f' + (c^2 d_1 + d_3 - c^4 - c d_2) f' .$$

Integrating the above equation once we get

$$\frac{1}{4} f''' = -\frac{3}{16} f^5 - 5 c f^4 + \left(\frac{d_1}{2} - 3 c^2\right) f^3 + \left(\frac{3}{2} c d_1 - 3 c^3 - \frac{3}{4} d_2\right) f^2$$

$$+ (c^2 d_1 + d_3 - c^4 - c d_2) f + d_4 ,$$

where $d_4$ is an integration constant. By using $f'$ as an integrating factor, we integrate once more. Finally, we get

$$(f')^2 = -\frac{1}{4} f^6 - 2 c f^5 + (d_1 - 6 c^3) f^4 + (4 c d_1 - 8 c^3 - 2 d_2) f^3$$

$$+ 4(c^2 d_1 + d_3 - c^4 - c d_2) f^2 + 8 d_4 f + 8 d_5 ,$$

where $d_5$ is an integration constant. If we apply the boundary conditions $f, f', f'', g, g', h, h', r, r' \to 0$ as $\xi \to \infty$, we get $d_1 = d_2 = d_3 = d_4 = d_5 = 0$. Hence the above equation becomes

$$(f')^2 = -\frac{1}{4} f^6 - 2 c f^5 + 6 c^2 f^4 - 8 c^3 f^3 - 4 c^4 f^2$$

$$= -\frac{f^2}{4} (f + 2 c)^4 .$$

Obviously, there is no real traveling wave solution of the case $\ell = 4$ with asymptotically vanishing boundary conditions.

Conjecture: For all even $\ell$, since we have the following equality:

$$(f')^2 = -\frac{f^2}{2c-2} (f + 2 c)^\ell , \quad u(x, t) = f(\xi) \quad \xi = x - ct , \quad (107)$$

the degenerate coupled KdV equation (2) does not have real traveling wave solution with asymptotically vanishing boundary conditions.

VI. GRAPHS OF THE EXACT SOLUTIONS

Here we give the graphs of exact solutions to see the behavior of the solutions (Figs. 2–10).

Cases 1(a) and 1(b) for $k = 1$:

According to the conditions on parameters, the parameters are chosen as

$$\alpha = \beta = 1 , \quad c = 2 , \quad d_1 = -7/4 , \quad d_2 = -7/2 , \quad d_3 = -3/2 .$$

Hence the solution becomes

$$u(x, t) = \text{sech}(\xi) - 2 , \quad \xi = x - 2t , \quad (108)$$

and the graph of this function is
Note that by the choice of the parameters of this case Eq. (5) becomes

\[ F(f) = -(f + 3)(f + 1)(f + 2)^2. \]

The numerical values of the zeros of \( F(f) \) are such that the graph corresponds to the exact solitary wave solution given in Sec. III C, part (a).

**Case 1(a) for different values of \( k \):**

Here to see the behavior of the solution by the change of the value of \( k \) we give the following graph:
Case 1(b) for $k = 0.5$:
The parameters are chosen as

$$
\alpha = \beta = 1, \quad c = 2, \quad d_1 = -\frac{25}{16}, \quad d_2 = -\frac{25}{8}, \quad d_3 = -\frac{39}{32}.
$$

The solution is

$$
\frac{d}{d\xi} = \frac{\xi - x}{\sin(\xi)}, \quad \xi = x - 2t,
$$

and the graph of this function is

Note that by the choice of the parameters of this case Eq. (5) becomes

$$
F(f) = -(f + 3)(f + 1)\left(f - (-2 + \frac{1}{2}\sqrt{3})\right)\left(f - (-2 - \frac{1}{2}\sqrt{3})\right).
$$

Since $F(f)$ has four different simple zeros, we expect periodic solution as in the graph.

Case 2(a) for $k = 0$:
The parameters are

$$
\alpha = -2, \quad \beta = 1, \quad c = 1, \quad d_1 = \frac{3}{4}, \quad d_2 = d_3 = 0, \quad \beta = 1.
$$

Hence the solution becomes

$$
\frac{d}{d\xi} = \frac{1}{-2 - \sqrt{3}\sin(\xi)}, \quad \xi = x - t,
$$

and the graph of this function is
Note that by the choice of the parameters of this case Eq. (5) becomes
\[ F(f) = -f^2(f - (-2 + \sqrt{3}))(f - (-2 - \sqrt{3})). \]
Here the function \( F(f) \) has one double zero \( f_2 = 0 \) and two simple zeros \( f_1 = -2 - \sqrt{3} \) and \( f_3 = -2 + \sqrt{3} \) so \( f_1 < f_3 < f_2 \). As it is stated in Sec. II B, part (4) we have periodic solution which can also be seen in the above graph.

Case 2(b) for \( k = 0 \): The parameters are chosen as
\[ a = 2, \quad b = -\sqrt{3}, \quad c = -1, \quad d_1 = \frac{3}{4}, \quad d_2 = d_3 = 0, \quad \beta = 1. \]
Hence the solution becomes
\[ u(x, t) = \frac{1}{2 - \sqrt{3} \cos(\xi)} \cdot \xi = x + t, \tag{111} \]
and the graph of this function is

FIG. 5. Graph of Case 2(a) \((k = 0)\).

FIG. 6. Graph of Case 2(b) \((k = 0)\).
Note that by the choice of the parameters of this case, Eq. (5) becomes
\[ F(f) = -f^2(f - (2 + \sqrt{3}))(f - (2 - \sqrt{3})). \]
The function \( F(f) \) has one double zero \( f_2 = 0 \) and two simple zeros \( f_1 = 2 - \sqrt{3} \) and \( f_3 = 2 + \sqrt{3} \) so \( f_2 < f_1 < f_3 \). As it is given in Sec. II B, part (4), the solution is periodic, which can be easily seen in the graph.

**Cases 2(b) and 2(c) for \( k = 1 \):** The parameters are chosen as
\[
\begin{align*}
  a &= 1, \quad b = -\sqrt{\frac{7}{8}}, \quad c = -\frac{9}{2}, \quad d_1 = 10, \quad d_2 = 4, \quad d_3 = -1, \quad \beta = \sqrt{7}.
\end{align*}
\]
Hence the solution becomes
\[
\begin{align*}
  u(x, t) = \frac{1}{1 - \sqrt{\frac{7}{8}} \operatorname{sech}(\sqrt{7} \xi)} & , \quad \xi = x + \frac{9}{2} t, \quad (112)
\end{align*}
\]

and the graph of this function is

Note that by the choice of the parameters of this case, Eq. (5) becomes
\[
F(f) = -(f - (8 + \sqrt{14}))(f - (8 - \sqrt{14}))(f - 1)^2.
\]
The numerical values of the zeros of \( F(f) \) are such that the graph corresponds to the exact solitary wave solution given in Sec. III C, part (a).

**Case 2(e) for \( k = 0 \):** The parameters are chosen as
\[
\begin{align*}
  a &= 1, \quad b = 2, \quad c = -\frac{1}{3}, \quad d_1 = \frac{5}{18}, \quad d_2 = -\frac{1}{6}, \quad d_3 = \frac{1}{24}, \quad \beta = \frac{2\sqrt{3}}{3}.
\end{align*}
\]
Hence the solution becomes
\[
\begin{align*}
  u(x, t) = \frac{\sin(\frac{2\sqrt{3}}{3} \xi)}{\sin(\frac{2\sqrt{3}}{3} \xi) + 2} & , \quad \xi = x + \frac{1}{3} t, \quad (113)
\end{align*}
\]

and the graph of this function is

FIG. 7. Graph of Cases 2(b)–2(c) \( (k = 1) \).
Note that by the choice of the parameters of this case, Eq. (5) becomes
\[ F(f) = -(f - 1)^2(f + 1)(f - \frac{1}{3}). \]
Here the function \( F(f) \) has one double zero \( f_2 = 1 \) and two simple zeros \( f_1 = -1 \) and \( f_3 = \frac{1}{3} \) so \( f_1 < f_3 < f_2 \). As it is noted in Sec. II B, part (4) we have periodic solution which can be seen in the graph.

**Case 2(f) for \( k = 0 \):** The parameters are chosen as
\[
\begin{align*}
    a &= 1, & b &= 2, & c &= -\frac{1}{3}, & d_1 &= \frac{5}{18}, & d_2 &= -\frac{1}{6}, & d_3 &= \frac{1}{24}, & \beta &= \frac{2\sqrt{3}}{3}.
\end{align*}
\]
Hence the solution becomes
\[ u(x, t) = \frac{\cos(2\sqrt{3}\xi)}{\cos(\frac{2\sqrt{3}}{3}\xi) + 2}, \quad \xi = x + \frac{1}{3}t, \] (114)
and the graph of this function is

![Graph of Case 2(e) (k = 0).](image)

![Graph of Case 2(f) (k = 0).](image)
Note that by the choice of the parameters of this case, Eq. (5) becomes
\[ F(f) = -(f - 1)^2(f + 1)(f - \frac{1}{3}). \]
The zeros of the function \( F(f) \) are same as in the previous case. So the graph fits to the fact given in Sec. II B, part (4).

**Case 2(f) for \( k = 1 \):** The parameters are chosen as
\[ a = 1, \quad b = 2, \quad c = \frac{1}{6}, \quad d_1 = \frac{1}{9}, \quad d_2 = d_3 = 0, \quad \beta = \sqrt{\frac{3}{3}}. \]
Hence the solution becomes
\[ u(x, t) = \frac{\text{sech}(\sqrt{\frac{3}{3}} \xi)}{\text{sech}(\sqrt{\frac{3}{3}} \xi) + 2}, \quad \xi = x - \frac{1}{6}t, \]
and the graph of this function is

Note that by the choice of the parameters of this case, Eq. (5) becomes
\[ F(f) = -f^4 - \frac{2}{3}f^3 + \frac{1}{3}f^2 = -(f - \frac{1}{3})(f + 1)f^2. \]
The numerical values of the zeros of \( F(f) \) are such that the graph corresponds to the exact solitary wave solution given in Sec. III C, part (a).

**VII. CONCLUSION**

We have studied symmetry reduced (traveling waves) equations of the Kaup-Boussinesq (KB) type of coupled degenerate KdV equations for \( \ell = 2 \). The reduced equation turns out to be such that the square of the derivative of the dependent variable is equal to a fourth degree polynomial of the dependent variable. There are four arbitrary constants in the polynomial function. We have investigated all possible cases and gave all solitary wave solutions which rapidly decay to some constants of the \((\ell = 2)\) KB equations. There are periodic solutions of this set of coupled KdV equations in terms of the Jacobi elliptic functions. We first introduced special solutions of this type where the zeros of \( F(f) \) satisfy certain constraints. If we remove these constraints among the zeros
we obtained the most general solution in terms of the elliptic functions of KB system under the assumed symmetry. There are four different such solutions which differ by the initial values at the origin. For illustration we have given the graphs of some interesting solutions. We have also initiated the work on the cases for $\ell = 3$ and $\ell = 4$. We have given some results concerning these cases. A detailed study of the traveling wave solutions of the cases $\ell = 3$ and $\ell = 4$ will be communicated later.

ACKNOWLEDGMENTS

This work is partially supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK).