1-) Suppose two spin \( \frac{1}{2} \) particles are known to be in the singlet configuration. Let \( S_a^{(1)} \) be the component of the spin angular momentum of particle number 1 in the direction defined by the unit vector \( \hat{a} \). Similarly, let \( S_b^{(2)} \) be the component of 2’s angular momentum in the direction \( \hat{b} \). Show that
\[
\langle S_a^{(1)} S_b^{(2)} \rangle = -\frac{\hbar^2}{4} \cos \theta
\]
where \( \theta \) is the angle between \( \hat{a} \) and \( \hat{b} \).

2-) The electron in a hydrogen atom occupies the combined spin and position state

\[
R_{21} \left( \sqrt{1/3} Y^0_1 \chi_+ + \sqrt{2/3} Y^1_1 \chi_- \right)
\]

a) If you measured the orbital angular momentum squared \( (L^2) \), what values might you get, and what is the probability of each?
b) Same for the \( z \) component of orbital angular momentum \( (L_z) \).
c) Same for the spin angular momentum squared \( (S^2) \).
d) Same for the \( z \) component of spin angular momentum \( (S_z) \).

Let \( \vec{J} = \vec{L} + \vec{S} \) be the total angular momentum.
e) If you measured the \( (J^2) \), what values might you get, and what is the probability of each?
f) Same for \( (J_z) \).
g) If you measured the position of the particle, what is the probability density for finding it at \( r, \theta, \phi \)??
h) If you measured both the \( z \) component of the spin and the distance from the origin (note that these are compatible observables), what is the probability density for finding the particle with spin up and at radius \( r \)?)

3-) Consider two spin \( \frac{1}{2} \) particles, whose spins are described by the Pauli operators \( \sigma_i \) and \( \sigma_j \). Let \( \hat{e} \) be the unit vector connecting the two particles and define the operator
\[
S_{12} = 3(\sigma_1 \cdot \hat{e})(\sigma_2 \cdot \hat{e}) - \sigma_1 \cdot \sigma_2
\]
Show that if the two particles are in an \( S = 0 \) state (singlet) then
\[
S_{12} \chi_{\text{singlet}} = 0
\]
Show that for a triplet state
\[
(S_{12} - 2)(S_{12} + 4) \chi_{\text{triplet}} = 0
\]