(1) 27-7-a) Multiply the equation $AB = 0$ by $A^{-1}$ from the left:

$$A^{-1}AB = 0 \iff B = 0$$

27-7-b) Since $A$ is not invertible, the system $AX = 0$ has a nontrivial solution. Let

$$B = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ x_2 & 0 & \cdots & 0 \\ \vdots & 0 & \cdots & 0 \\ x_n & 0 & \cdots & 0 \end{pmatrix}$$

where $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \neq 0$ is the nontrivial solution of the system $AX = 0$. Then $AB = 0$.

(2) 27-8 ($\implies$) Since it is invertible, the matrix $A$ is row equivalent to identity matrix. We have 2 cases: $ac = 0$ or $ac \neq 0$.

**Case 1: $ac=0$** Since $A \sim I_2$, they can not be both zero. Without loss of generality, we can assume $c = 0$ and $a \neq 0$. In this case, we have

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

This means that $d \neq 0$ and hence $ad - bc = ad \neq 0$.

**Case 2: $ac \neq 0$** Assume for a contradiction that $ad - bc = 0$. Then,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & \frac{da - bc}{a} \end{pmatrix}.$$  

Contradiction since the reduced row echelon form of $A$ is the identity matrix.

($\iff$) **Case 1: $ac=0$** Since $ad - bc \neq 0$, either $a = 0$ or $c = 0$. WLOG, let $c = 0$. In this case, $ad \neq 0$ i.e $a \neq 0$ and $d \neq 0$. Therefore, we have

$$A = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & d \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$  

i.e $A$ is invertible.

**Case 2: $ac \neq 0$**
\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} 1 & b \\ 0 & \frac{da-bc}{d} \end{pmatrix} \sim \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\]
i.e \( A \) is invertible.

(3) **34-6** We need to prove that the operations are well-defined and \( V \) with these operations satisfies the vector space axioms. The latter is follows from the properties of addition and multiplication of real numbers. For the first one, note that

\[
(f + g)(-t) = f(-t) + g(-t) = \overline{f(t)} + \overline{g(t)} = \overline{(f + g)(t)}, \text{i.e. } f + g \in V
\]

\[
(cf)(-t) = cf(-t) = \overline{cf(t)} = \overline{c\overline{f(t)}} = cf(t), \text{ i.e. } cf \in V (\text{since } c \in \mathbb{R}).
\]

(4) **39-4** Since

\[
\begin{pmatrix} 2 & -1 & \frac{4}{3} & -1 & 0 \\ 1 & 0 & \frac{2}{3} & 0 & -1 \\ 9 & -3 & 6 & -3 & -3 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & \frac{2}{3} & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},
\]

then the space of solutions of \( AX = 0 \) is

\[
\left\{ \left( \frac{-2a}{3} + c, -b + 2c, a, b, c \right) \mid a, b, c \in \mathbb{R} \right\}.
\]

Therefore, the set \( \left\{ \left( \frac{-2}{3}, 0, 1, 0, 0 \right), (0, -1, 0, 1, 0), (1, 2, 0, 0, 1) \right\} \) spans the space of solutions of \( AX = 0 \).

(5) **40-9** Since \( W_1 + W_2 = V \), for every \( v \in V \) there exists \( \alpha_1 \in W_1 \) and \( \alpha_2 \in W_2 \) such that \( v = \alpha_1 + \alpha_2 \). This proves the existence. For the uniqueness, let \( \alpha_1, \alpha'_1 \in W_1 \) and \( \alpha_2, \alpha'_2 \in W_2 \) be such that

\[
v = \alpha_1 + \alpha_2 = \alpha'_1 + \alpha'_2 \iff \underbrace{\alpha_1 - \alpha'_1}_{\in W_1} = \underbrace{\alpha'_2 - \alpha_2}_{\in W_2}.
\]

Since \( W_1 \cap W_2 = \{0\} \), we have \( \alpha_1 = \alpha'_1 \) and \( \alpha_2 = \alpha'_2 \).