EQUILIBRIUM CANTOR-TYPE SETS

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Abstract. Equilibrium Cantor-type sets are suggested. This allows to obtain Green functions with various moduli of continuity and compact sets with preassigned growth of Markov’s factors.

1. Introduction

If a compact set $K \subset \mathbb{C}$ is regular with respect to the Dirichlet problem then the Green function $g_{\mathbb{C}\setminus K}$ of $\mathbb{C}\setminus K$ with pole at infinity is continuous throughout $\mathbb{C}$. We are interested in analysis of a character of smoothness of $g_{\mathbb{C}\setminus K}$ near the boundary of $K$. For example, if $K \subset \mathbb{R}$ then the monotonicity of the Green function with respect to the set $K$ implies that the best possible behavior of $g_{\mathbb{C}\setminus K}$ is $\text{Lip}^{1/2}$ smoothness. An important characterization for general compact sets with $g_{\mathbb{C}\setminus K} \in \text{Lip}^{1/2}$ was found in [17] by V.Totik. The monograph [17] revives interest in the problem of boundary behavior of Green functions. Various conditions for optimal smoothness of $g_{\mathbb{C}\setminus K}$ in terms of metric properties of the set $K$ are suggested in [7], and in papers by V.Andrievskii [2]-[3]. On the other hand, compact sets are considered in [1], [8] such that the corresponding Green functions have moduli of continuity equal to some degrees of $h$, where the function $h(\delta) = (\log \frac{1}{\delta})^{-1}$ defines the logarithmic measure of sets. For a recent result on smoothness of $g_{\mathbb{C}\setminus K_0}$, where $K_0$ is the classical Cantor set, see [13].

Here the Cantor-type set $K(\gamma)$ is constructed as the intersection of the level domains for a certain sequence of polynomials depending on the parameter $\gamma = (\gamma_n)_{n=1}^{\infty}$ (Section 2). In favor of $K(\gamma)$, in comparison to usual Cantor-type sets, it is equilibrium in the following sense.

Let $\lambda_s$ denote the normalized Lebesgue measure on the closed set $E_s$, where $K(\gamma) = \bigcap_{s=0}^{\infty} E_s$. Then $\lambda_s$ converges in the weak* topology to the equilibrium measure of $K(\gamma)$ (Section 5). This is not valid for geometrically symmetric, though very small Cantor-type sets with positive capacity.

Different values of $\gamma$ provide a variety of the Green functions with diverse moduli of continuity (Section 7).

In Section 8 we estimate Markov’s factors for the set $K(\gamma)$ and construct a set with preassigned growth of subsequence of Markov’s factors.

In Section 9 a set $K(\gamma)$ is presented such that the Markov inequality on $K(\gamma)$ does not hold with the best Markov’s exponent $m(K(\gamma))$. This gives an affirmative answer to the problem (5.1) in [4].

For basic notions of logarithmic potential theory we refer the reader to [10], [12], and [15].

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We use the notation $| \cdot |_K$ for the supremum norm on $K$, log denotes the natural logarithm, $0 \cdot \log 0 := 0$.

2. Construction of $K(\gamma)$

Suppose we are given a sequence $\gamma = (\gamma_s)_{s=1}^{\infty}$ with $0 < \gamma_s < 1/4$. Let $r_0 = 1$ and $r_s = \gamma_s r_{s-1}^2$ for $s \in \mathbb{N}$. We define inductively a sequence of real polynomials: let $P_2(x) = x(x-1)$ and $P_{2s+1} = P_{2s}(P_{2s} + r_s)$ for $s \in \mathbb{N}$. It is easy to check by induction that the polynomial $P_{2s}$ has $2^{s-1}$ points of minimum with equal values $P_{2s} = -r_{s-1}^2/4$.

By that we have a geometric procedure to define new (with respect to $P_{2s}$) zeros of $P_{2s+1}$: they are abscessas of points of intersection of the line $y = -r_s$ with the graph $y = P_{2s}$. Let $E_s$ denote the set $\{ x \in \mathbb{R} : P_{2s+1}(x) \leq 0 \}$. Since $r_s < r_{s-1}^2/4$, the set $E_s$ consists of $2^s$ disjoint closed basic intervals $I_{j,s}$. In general, the lengths $l_{j,s}$ of intervals of the same level are different, however, by the construction of $K(\gamma)$, we have $\max_{1\leq j \leq 2^s} l_{j,s} \to 0$ as $s \to \infty$. Clearly, $E_{s+1} \subset E_s$. Set $K(\gamma) = \cap_{s=0}^{\infty} E_s$.

Let us show that the sequence of level domains $D_s = \{ z \in \mathbb{C} : |P_{2s}(z) + r_s/2| < r_s/2 \}$, $s = 1, 2, \cdots$, is a nested family.

Lemma 1. Given $z \in \mathbb{C}$ and $s \in \mathbb{N}$, let $w_s = 2 r_s^{-1} P_{2s}(z) + 1$. Suppose $|w_s| = 1 + \varepsilon$ for some $\varepsilon > 0$. Then $|w_{s+1}| > 1 + 4 \varepsilon$.

Proof: We have $w_{s+1} = (2\gamma_s + 1)(w_s^2 - 1 + 2\gamma_s + 1)$. Therefore, $|w_{s+1}|$ attains its minimal value if $w_s \in \mathbb{R}$, so $|w_{s+1}| \geq (2\gamma_s + 1)(2\varepsilon + \varepsilon^2 + 2\gamma_s + 1) > 1 + \varepsilon^2 + 1 + 4 \varepsilon$. \qed

Theorem 1. We have $D_s \setminus K(\gamma)$.

Proof: The embedding $\overline{D}_{s+1} \subset \overline{D}_s$ is equivalent to the implication

$$|P_{2s}(z) + r_s/2| > r_s/2 \implies |P_{2s+1}(z) + r_{s+1}| > r_{s+1}/2,$$

which we have by Lemma 1.

For each $j \leq 2^s$ the real polynomial $P_{2s}$ is monotone on $I_{j,s}$ and takes values 0 and $-r_s$ at its endpoints. Therefore, $E_s \subset \overline{D}_s$ and $K(\gamma) \subset \cap_{s=0}^{\infty} \overline{D}_s$.

For the inverse embedding, let us fix $z \notin K(\gamma)$. We need to find $s$ with $z \notin \overline{D}_s$. Suppose first $z \in \mathbb{R}$. Since $\overline{D}_s \cap \mathbb{R} = E_s$, the condition $z \notin E_s$ gives the desired $s$.

Let $z = x + iy$ with $y \neq 0, x \notin K(\gamma)$. By the above, $x \notin \overline{D}_s$ for some $s$. All zeros $(x_j)_{j=1}^{2^s}$ of the polynomial $P_{2s}$ are real. Therefore, $|P_{2s}(z) + r_s/2| > |P_{2s}(x) + r_s/2| > r_s/2$ and $z \notin \overline{D}_s$.

It remains to consider the case $z = x + iy$ with $y \neq 0, x \in K(\gamma)$. There is no loss of generality in assuming $|y| < 2$. Let us fix $s$ with $\max_{1\leq j \leq 2^s} l_{j,s} < y^2/2$ and $k$ with $x \in I_{k,s} = [a,b]$. Here, $|P_{2s}(a) + r_s/2| = r_s/2$. Let us show that $|P_{2s}(z) + r_s/2| > |P_{2s}(a) + r_s/2|$ by comparison the distances from $z$ and from $a$ to the zero $x_j$.

If $j < k$ then $|a - x_j| \leq |a - x|$, which is less than the hypotenuse $|z - x_j|$.

If $j = k$ then $|a - x_k| \leq l_{k,s} < y^2/2 < |y| \leq |z - x_k|$.

If $j > k$ then $|a - x_j| = x_j - b + l_{k,s}$. Therefore, $|a - x_j|^2 < |x_j - b|^2 + 2 l_{k,s} < |x_j - b|^2 + y^2 \leq |z - x_j|^2$. \qed
Corollary 1. The set \( K(\gamma) \) is polar if and only if \( \lim_{s \to \infty} 2^{-s} \log \frac{2}{r_s} = \infty \). If this limit is finite and \( z \notin K(\gamma) \), then
\[
g_{C \setminus K(\gamma)}(z) = \lim_{s \to \infty} 2^{-s} \log |P_{2s}(z)/r_s|.
\]

Proof: Clearly, \( g_{C \setminus K_s}(z) = 2^{-s} \log |2r_s^{-1} P_{2s}(z) + 1| \). The sequence of the corresponding Robin constants \( \text{Rob}(D_s) = 2^{-s} \log \frac{2}{r_s} \) increases. If its limit is finite, then, by the Harnack Principle (see e.g. [15], Th.0.4.10), \( g_{C \setminus K_s} \neq g_{C \setminus K(\gamma)} \) uniformly on compact subsets of \( C \setminus K(\gamma) \). Suppose \( z \notin K(\gamma) \). Then for some \( q \in \mathbb{N} \) and \( \varepsilon > 0 \) we have \( |w_q| = 1 + \varepsilon \). By Lemma 1, \( |w_s| > 1 + 4^{s-q} \varepsilon \), so, for large \( s \), the value \( |P_{2s}(z)/r_s| \) dominates 1. This gives the desired representation of \( g_{C \setminus K(\gamma)} \).

The next corollary is a consequence of the Kolmogorov criterion (see e.g. [9], T.3.2.1). Recall that a monic polynomial \( Q_n \) is a Chebyshev polynomial for a compact set \( K \) if the value \( |Q_n|_K \) is minimal among all monic polynomials of degree \( n \).

Corollary 2. The polynomial \( P_{2s} + r_s/2 \) is the Chebyshev polynomial for the set \( K(\gamma) \).

Example 1. Let us consider the limit case, when \( \gamma_s = 1/4 \) for all \( s \), so \( r_s = 4^{1-2s} \).

3. Location of zeros

We decompose all zeros of \( P_{2s} \) into \( s \) groups. Let \( X_0 = \{x_1, x_2\} \). \( X_1 = \{x_3, x_4\} = \{l_{1,1}, 1-l_{2,1}\}, \ldots, X_k = \{x_{2k+1}, \ldots, x_{2k+1}\} = \{l_{1,k}, l_{1,k-1}-l_{2,k}, \ldots, 1-l_{2,k}\} \), so \( X_k = \{x : P_{2s}(x) + r_k = 0\} \) contains all zeros of \( P_{2s} \) that are not zeros of \( P_{2k} \).

Set \( Y_s = \bigcup_{k=0}^{s} X_k \). Then \( P_{2s}(x) = \prod_{x_k \in Y_{s-1}}(x-x_k) \). Since \( P_{2s} = P_{2s-1}(2P_{2s-1} + r_{s-1}) \) for \( s \geq 2 \), we have
\[
P'_{2s}(y) = r_{s-1} P'_{2s-1}(y), \quad y \in Y_{s-2}; \quad P'_{2s}(x) = -r_{s-1} P'_{2s-1}(x), \quad x \in X_{s-1}.
\]

After iteration this gives
\[
|P'_{2s}(x)| = r_{s-1} r_{s-2} \cdots r_q |P'_{2s}(x)| \quad \text{for} \quad x \in X_q \quad \text{with} \quad q < s.
\]

From here, for example, \( |P'_{2s}(0)| = r_{s-1} r_{s-2} \cdots r_1 \).

The identity \( P_{2s+1}(y) = P_{2s}(y) \prod_{x_k \in X_s} (y-x_k) = P_{2s}(y) (P_{2s}(y) + r_s) \) implies \( P_{2s}(y) + r_s = \prod_{x_k \in X_s} (y-x_k) \). Thus,
\[
\prod_{x_k \in X_s} (y-x_k) = r_s \quad \text{for} \quad y \in Y_{s-1}.
\]

Our next goal is to express the values of \( x_k \in X_s \) in terms of the function \( u(t) = \frac{1}{2} - \frac{1}{2} \sqrt{1-4t} \) with \( 0 < t < \frac{1}{4} \). Clearly, \( u(t) \) and \( 1-u(t) \) are the solutions of the equation \( P_2(x) + t = 0 \). Let us consider the expression
\[
x = f_1(\gamma_1 \cdot f_2(\gamma_2 \cdots f_{s-1}(\gamma_{s-1} \cdot f_s(\gamma_s))) \cdots),
\]
where \( f_k = u \) or \( f_k = 1-u \) for \( 1 \leq k \leq s \), so \( f_k(t)(1-f_k(t)) = t \). We have \( P_2(x) = -\gamma_1 \cdot f_2(\gamma_2 \cdots) \) with \( \gamma_1 = r_1 \). Hence, \( P_4(x) = P_2(x)(P_2(x) + r_1) = -r_1^2 f_2(1-
\[ f_2 = -r_1^2 \gamma_2 f_3 = -r_2 f_3 (\gamma_3 \cdots). \] We continue in this fashion to obtain eventually \[ P_{2^s}(x) = -r_{2^{s-1}} \gamma_s = -r_s, \] which gives \( x \in X_s. \)

The formula (4) provides \( 2^s \) possible values \( x. \) Let us show that they are all different, so any \( x_k \in X_s \) can be represented by means of (4). Since \( u \) increases and \( u(a) < 1 - u(b) \) for \( a, b \in (0, \frac{1}{3}) \), we have \( u(\gamma_1 \cdot u(\gamma_2 \cdots u(a)) \cdots) < u(\gamma_1 \cdot u(\gamma_2 \cdots u(1 - u(b))) \cdots). \) In general, let \( x_i = u(\gamma_1 \cdot u(\gamma_2 \cdots u(1 - u(a)) \cdots) \) and \( x_j = u(\gamma_1 \cdot u(\gamma_2 \cdots u(1 - u(a)) \cdots) \), that is the first \( k_m \) functions \( f_k \) for both points are identical, whereas \( f_{k+1} = 1 - u \) for \( x_i \) and \( u \) for \( x_j \). The straightforward comparison shows that \( x_i > x_j \) for odd \( m \) and \( x_i < x_j \) otherwise.

**Lemma 2.** Let \( s \in \mathbb{N} \) and \( 1 \leq j \leq 2^s \). Then \( l_{1,s} \leq l_{j,s}. \)

**Proof:** Assume without loss of generality that \( j \) is odd. Then \( I_{j,s} = [y, x] \) with \( x \in X_s, y \in X_m \) where \( 1 \leq m \leq s - 1 \). The case \( m = 0 \) can be excluded, since then \( y \in \{0, 1\} \) and \( l_{1,s} = \tau_k \leq t_k \). Therefore, \( |f_k(t)\| = \frac{1}{\sqrt{1 - 4t}} \geq \frac{1}{\sqrt{4 - 4t}} = u_k(\tau_k). \) On the other hand, \( u(t) \sqrt{1 - 4t} < t \) for \( 0 < t < \frac{1}{4} \), as is easy to check. This gives \( |F'(\xi)| = |f'(1)| \cdot \gamma_1 \cdots f_m(\xi) \cdot \gamma_{m+1} \cdots u(\gamma_m \cdots u(\gamma_s)) \) for \( 1 \leq k \leq m - 1 \). By the above, \( \tau_k \leq t_k \). Therefore, \( l_{j,s} > u(\gamma_1 \cdots u(\gamma_{m+1} \cdots u(\gamma_s))) \) and \( \tau_m = \gamma_m \cdot u(\gamma_{m+1} \cdots u(\gamma_s)) \) for \( k \leq m - 2 \) and \( \tau_{m-1} = \gamma_{m-1} \cdot u(\xi) \), we obtain \( |F'(\xi)| > \frac{u(\gamma_1)}{\xi} \) and

\[ l_{j,s} > u(\gamma_1 \cdots u(\gamma_{m+1} \cdots u(\gamma_s))) \cdot \frac{u(\gamma_1)}{\xi} \cdot \gamma_m \cdot u(\gamma_{m+1} \cdots u(\gamma_s)) \cdots). \]

Taking into account the representation \( u(t) = \frac{2^t}{1 + \sqrt{1 - 4t}} \), we have \( u(\alpha t) < \alpha u(t) \) for \( 0 < \alpha < 1 \). The value \( \alpha = \frac{1}{\xi} \cdot \gamma_m \cdot u(\gamma_{m+1} \cdots u(\gamma_s)) \) satisfies this condition. Therefore, \( l_{1,s} = u(\gamma_1 \cdot u(\gamma_2 \cdots u(\gamma_{m+1} \cdots u(\gamma_s))) \cdots) \) is the desired conclusion. \( \square \)

### 4. Auxiliary results

From now on we make the assumption

\[ \gamma_s \leq 1/32 \quad \text{for} \quad s \in \mathbb{N}. \] (5)

Each \( I_{j,s} \) contains two adjacent basic subintervals \( I_{2j-1,s+1} \) and \( I_{2j,s+1} \). Let \( h_{j,s} = l_{j,s} - l_{2j-1,s+1} - l_{2j+1,s+1} \) be the distance between them.

**Lemma 3.** Suppose \( \gamma \) satisfies (5). Then the polynomial \( P_{2^s} \) is convex on \( I_{j,s-1} \) and \( l_{2j-1,s} + l_{2j,s} < 4 \gamma_s l_{j,s-1} \) for \( 1 \leq j \leq 2^{s-1} \). Thus, \( h_{j,s} > \frac{7}{8} l_{j,s} \) for \( s \geq 0, 1 \leq j \leq 2^s \).

**Proof:** We proceed by induction. If \( s = 1 \) then \( P_2 \) is convex on \( I_{1,0} = [0, 1] \). Let us show that \( l_{1,1} + l_{2,1} < 4 \gamma_1 \). The triangle \( \Delta \) with the vertices \((0, 0), (1, 0), (\frac{1}{2}, -\frac{1}{4})\) is entirely situated in the epigraph \( \{(x, y) \in \mathbb{R}^2 : P_2(x) \leq y \} \). The line \( y = -t_1 \) intersects \( \Delta \) along the segment \([A, B] \). By convexity of \( P_2 \), we have \( h_{1,0} = 1 - l_{1,1} - l_{2,1} > |B - A|. \)
The triangle $\Delta_1$ with the vertices $A, B, (\frac{1}{2}, -\frac{1}{4})$ is similar to $\Delta$. Therefore, $\frac{1}{4} |B - A| = \frac{1}{4} - r_1$. Here, $r_1 = \gamma_1$, and the result follows.

Suppose we have convexity of $P_2^s|_{I_{j,s-1}}$ and the desired inequalities for $k = 1, 2, \ldots, s - 1$. Fix $j \leq 2^{s-1}$ and $x \in I_{j,s-1} = [a, b]$. Then $P_2^s(x) = (x - a)(x - b)g(x)$, where $g(x) = \prod_{k=1}^n(x - z_k)$ with $n = 2^s - 2$. Hence,

$$P_2^s(x) = g(x) \left[ 2 + 2 \sum_{k=1}^n \frac{2x - a - b}{x - z_k} + \sum_{k=1}^n \sum_{i \neq k} \frac{(x - a)(x - b)}{(x - z_k)(x - z_i)} \right].$$

Clearly, the polynomial $g$ is positive on $I_{j,s-1}$, $|2x - a - b| \leq l_{j,s-1}$, and $|(x - a)(x - b)| \leq \frac{1}{4} l_{j,s-1}^2$. For convexity of $P_2^s|_{I_{j,s-1}}$ we only need to check that $8 \geq 8 l_{j,s-1} - 2 + l_{j,s-1}^2 \sum_{i \neq k} |x - z_k|^{-1} < \frac{4}{1 - 4\gamma_{s-1}} l_{j,s-1}^2 \sum_{i \neq k} |x - z_k|^{-1}$.

Let us consider the basic intervals containing $x$: $I_{j,s-1} \subset I_{m,s-2} \subset I_{q,s-3} \subset \cdots \subset I_{1,0}$. The interval $I_{m,s-2}$ contains two zeros of $g$. For them $|x - z_k| \geq h_{m,s-2} > (1 - 4\gamma_{s-1}) l_{m,s-2}$ and $l_{j,s-1} \sum_{i \neq k} |x - z_k|^{-1} < \frac{4\gamma_{s-1}}{1 - 4\gamma_{s-1}}$, by inductive hypothesis. The last fraction does not exceed $1/7$. Similarly, $I_{q,s-3}$ contains another four zeros of $g$ with $l_{j,s-1} \sum_{i \neq k} |x - z_k|^{-1} < \frac{1}{7} \cdot \frac{1}{8}$. We continue in this fashion to obtain $l_{j,s-1} \sum_{i \neq k} |x - z_k|^{-1} < \frac{1}{7} \cdot \frac{1}{8} \cdots < \frac{1}{7^s}$. In the same way, $l_{j,s-1} \sum_{i \neq k} |x - z_k|^{-1} |x - z_i|^{-1} < \frac{1}{7^s} \frac{1}{8^s}$, which gives $P_2^s|_{I_{j,s-1}} > 0$. Arguing as above, by means of convexity of $P_2^s|_{I_{j,s-1}}$, it is easy to show the second statement of Lemma. □

Let $\delta_s = \gamma_1 \gamma_2 \cdots \gamma_s$, so $r_1 r_2 \cdots r_{s-1} \delta_s = r_2$.

**Lemma 4.** If $\gamma$ satisfies (5) then for any $x_k \in Y_{s-1}$ with $s \in \mathbb{N}$

$$\exp(-16 \sum_{k=1}^s \gamma_k) \cdot r/s \cdot \delta_s < |P_2^s(x_k)| < |P_2^s|_{E_s} = r_2/\delta_s$$

and

$$\delta_s < l_{i,s} \exp(16 \sum_{k=1}^s \gamma_k) \cdot \delta_s \quad \text{for} \quad 1 \leq i \leq 2^s.$$

**Proof:** From (2) it follows that $|P_2^s|_{E_s} \geq |P_2^s(0)| = r_2/\delta_s$. In order to get the corresponding lower bound, let us fix $I_{s,s} \subset E_s$. Without loss of generality we can assume that $i = 2j - 1$ is odd. Then $I_{s,s} \subset I_{j,s-1}$ and $I_{s,s} = [y, x]$ with $y \in Y_{s-1}, x = y + l_{i,s} \in X_s$. By Lemma 3, $|P_2^s|_{E_s}$ decreases on $[y, x]$, so $|P_2^s(x)| < |P_2^s(y)|$. We will estimate $|P_2^s(x)|$ from below in terms of $|P_2^s(y)|$.

The point $x$ is a zero of $P_{2s+1}$. Therefore, $P_{2s+1}(x) = (x - y) \cdot \prod_{y_k \in Y_s} |x - y_k| = (x - y) \cdot \prod_{y_k \in Y_s} |y - y_k| \cdot \beta$, where $Y_s = Y_s \setminus \{x, y\}$, $\beta = \prod_{y_k \in Y_s} (1 + \frac{l_{i,s}}{y - y_k})$. Here, $(x - y) \cdot \prod_{y_k \in Y_s} |y - y_k| = \prod_{x_k \in X_s} |y - x_k| \prod_{y_k \in Y_{s-1}, y_k \neq y} |y - y_k| = r_s |P_2^s(y)|$, by (3). On the other hand, by (1), $P_{2s+1}(x) = r_s |P_2^s(y)|$. Hence, $|P_2^s(x)| = \beta |P_2^s(y)|$. Let us estimate $\beta$ from below. We can take into account only $y_k \in Y_s$ with $y_k > y$, since otherwise the corresponding term in $\beta$ exceeds 1. The interval $I_{j,s-1}$ contains two points $y_k$ with $y_k - y > h_{j,s-1}$. Lemma 3 yields $1 + \frac{l_{i,s}}{y - y_k} > 1 - \frac{8}{7} \cdot l_{i,s} = 1 - \frac{8}{7} \cdot \gamma_{s-1}$. Therefore, $\beta > l_{i,s} \beta_1 \cdot \beta_2 \cdot \beta_3 \cdots \beta_{s-1} \cdot \beta_s$, where $\beta_1 = \prod(1 + l_{i,s})$, $\beta_2 = \prod_{1 < i \leq s} (1 + \frac{l_{i,s}}{y - y_k})$. Hence, $|P_2^s(x)| = \beta |P_2^s(y)|$. Let us estimate $\beta$ from below. We can take into account only $y_k \in Y_s$ with $y_k > y$, since otherwise the corresponding term in $\beta$ exceeds 1. The interval $I_{j,s-1}$ contains two points $y_k$ with $y_k - y > h_{j,s-1}$. Lemma 3 yields $1 + \frac{l_{i,s}}{y - y_k} > 1 - \frac{8}{7} \cdot l_{i,s} = 1 - \frac{8}{7} \cdot \gamma_{s-1}$.
For the next four points (let \( I_{j,s-1} \subset I_{m,s-2} \)) we have \( y_k - y > h_{m,s-2} \) and \( 1 + \frac{h_{s-1}}{y - y_k} > 1 - \frac{8}{7} \cdot \frac{t_{s-1}}{h_{m,s-2}} > 1 - \frac{8}{7} \cdot 4\gamma_s \cdot 4\gamma_{s-1} \geq 1 - \frac{1}{7} \cdot 4\gamma_s \), by (5).

We continue in this fashion obtaining \( \log \beta > \sum_{k=1}^s 2^k \log(1 - \frac{4}{7} \cdot 8^{2-k} \gamma_s) \). If \( 0 < a < \frac{1}{7} \) then \( \log(1 - a) > 4a \log \frac{3}{4} > -1.16a \). A straightforward calculation shows that \( \log \beta > -16 \gamma_s \). Thus,

\[
\exp(-16 \gamma_s) \left| P_{2s}'(y) \right| > \left| P_{2s}'(x) \right| > \left| P_{2s}'(y) \right|.
\]

Combining this inequality with (2) yields the first statement of Lemma. Indeed, let \( x = l_{i_1,m_1} - l_{i_2,m_2} + \cdots - l_{i_{q-1},m_{q-1}} + l_{i_q,m_q} \) with \( 1 \leq m_1 < \cdots < m_q = s \). Then \( y \in X_{m_q-1} \). We use (6), then (2) for \( y \), then (6) with \( y \) instead of \( x \) and \( z = l_{i_1,m_1} - l_{i_2,m_2} + \cdots + l_{i_{q-2},m_{q-2}} \in X_{m_{q-2}} \) instead of \( y \), then (2) for \( z \), etc. Finally,

\[
\exp(-16(\gamma_{m_1} + \cdots + \gamma_{m_q})) r_1 r_2 \cdots r_{s-1} < \left| P_{2s}'(x) \right| < r_1 r_2 \cdots r_{s-1}.
\]

If \( m_k = k \) for \( 1 \leq k \leq s \), then all \( \gamma_k \) are present in the corresponding sum. Monotonicity of \( \left| P_{2s}' \right| \) on \( [y, x] \) gives the desired conclusion.

The second statement of Lemma can be obtained by the Mean Value Theorem, since \( P_{2s}(y) = 0 \), \( P_{2s}(y + l_{i,s}) = -r_s \). In particular, (6) with \( x = l_{1,s}, y = 0 \) yields

\[
\delta_s < l_{1,s} < \delta_s \cdot e^{16 \gamma_s} < 2 \delta_s.
\]

\( \square \)

A.F.Beardon and Ch.Pommerenke introduced in [6] the concept of uniformly perfect sets. A dozen of equivalent descriptions of such sets are suggested in [10, p. 343]. We use the following: a compact set \( K \subset \mathbb{C} \) is uniformly perfect if \( K \) has at least two points and there exists \( \varepsilon_0 > 0 \) such that for any \( z_0 \in K \) and \( 0 < r \leq \text{diam}(K) \) the set \( K \cap \{z : \varepsilon_0 r < |z - z_0| < r\} \) is not empty.

**Theorem 2.** The set \( K(\gamma) \), provided (5), is uniformly perfect if and only if \( \inf \gamma_s > 0 \).

**Proof:** Suppose \( K(\gamma) \) is uniformly perfect. The values \( z_0 = 0 \) and \( r = l_{1,s-1} - l_{2,s} \) in the definition above imply \( l_{1,s} + l_{2,s} > \varepsilon_0 l_{1,s-1} \). By Lemma 3, we have \( 4\gamma_s > \varepsilon_0 \), so \( \inf \gamma_s \geq \varepsilon_0/4 \), which is our claim.

For the converse, assume \( \gamma_s \geq \gamma_0 > 0 \) for all \( s \). Let us show that \( l_{i,s} > \frac{1}{2} \gamma_0 l_{j,s-1} \) for any intervals \( I_{i,s} \subset I_{j,s-1} \), which clearly gives uniform perfectness of \( K(\gamma) \). Fix \( I_{i,s} \subset I_{j,s-1} \). Let \( x, y \) be the endpoints of \( I_{i,s} \) with \( x \in X_s, y \in Y_{s-1} \).

Suppose first that \( y \in X_{s-1} \). By the Mean Value Theorem, \( l_{i,s} \left| P_{2s}'(\xi) \right| = r_s \) for some \( \xi \in I_{i,s} \). By the monotonicity of \( \left| P_{2s}' \right| \) on \( I_{i,s} \), we have \( \left| P_{2s}'(\xi) \right| < \left| P_{2s}'(y) \right| \), which is \( r_{s-1} \left| P_{2s-1}'(y) \right| \), by (1). Here, \( \left| P_{2s-1}'(y) \right| < \left| P_{2s-1}'(z) \right| \), where \( z \in Y_{s-2} \) is another endpoint of \( I_{j,s-1} \). Therefore, \( l_{i,s} > \gamma_s r_{s-1} / \left| P_{2s-1}'(z) \right| \). On the other hand, \( l_{j,s-1} = r_{s-1} / \left| P_{2s-1}'(\eta) \right| \) with \( \eta \in I_{j,s-1} \), so \( \left| P_{2s-1}'(\eta) \right| > \left| P_{2s-1}'(z) \right| / e^{16 \gamma_{s-1}} \), by (6). Hence, \( l_{i,s} > \gamma_s l_{j,s-1} e^{16 \gamma_{s-1}} \geq \frac{1}{2} \gamma_0 l_{j,s-1} \).

The case \( y \in Y_{s-2} \) is very similar. Here at once \( y \) plays the role of \( z \). \( \square \)
5. $K(\gamma)$ is equilibrium

Here and in the sequel we consider $r_s$ in the form $r_s = 2 \exp(-R_s \cdot 2^s)$. Recall that $R_s$ is the Robin constant for $\mathcal{D}_s$ and $R_s \uparrow R$, which is finite if $K(\gamma)$ is not a polar set. In this case, let $\rho_s = R - R_s$. Since $r_0 = 1$, we have $\rho_0 = R - \log 2$. Clearly, $\gamma_s = \frac{1}{2} \exp[2^s(\rho_s - \rho_{s-1})]$ and $\delta_s = 2^{-s} \exp(2^s \rho_s - \sum_{k=1}^{s-1} 2^k \rho_k - 2 \rho_0)$. From this,

$$2^{-s} \log \delta_s \to 0 \quad \text{as} \quad s \to \infty. \quad (8)$$

Given $s \in \mathbb{N}$, let us uniformly distribute the mass $2^{-s}$ on each $I_{j,s}$ for $1 \leq j \leq 2^s$. We will denote by $\lambda_s$ the normalized (in this sense) Lebesgue measure on the set $E_s$, so $d\lambda_s = (2^s l_{j,s})^{-1} dt$ on $I_{j,s}$.

If $\mu$ is a finite Borel measure of compact support then its logarithmic potential is defined by $U^\mu(z) = \int \log \frac{1}{|z-t|} d\mu(t)$. We will denote by $\mu_K$ the equilibrium measure of $K$, $\rightharpoonup$ means convergence in the weak* topology.

Let $I = [a, b]$ with $b - a \leq 1$, $z \in I$. By partial integration,

$$\int_I \log \frac{1}{|z-t|} dt = b - a - (z-a) \log(z-a) - (b-z) \log(b-z).$$

It follows that

$$(b-a) \log \frac{e}{b-a} < \int_I \log \frac{1}{|z-t|} dt < (b-a) \log \frac{2e}{b-a}. \quad (9)$$

**Lemma 5.** Let $\gamma$ satisfy (5) and $R < \infty$. Then $U^{\lambda_s}(z) \to R$ for $z \in K(\gamma)$ as $s \to \infty$.

**Proof:** Fix $z \in K(\gamma)$. Given $s$, let $z \in I_{j,s}$ for $1 \leq j \leq 2^s$. From (9) we have $\int_{I_{j,s}} \log |z-t|^{-1} d\lambda_s(t) < 2^{-s} (2 + \log l_{j,s}^{-1})$, which is $o(1)$ as $s \to \infty$, by Lemma 4 and (8).

To estimate $\sum_{k=1, k \neq j}^{2^s} \int_{I_{k,s}} \log |z-t|^{-1} d\lambda_s(t)$ we use $P_{2^s}(x) = \prod_{k=1}^{2^s} (x - y_k)$ with $y_k \in I_{k,s}$. As above, let $I_{j,s} \subset I_{m,s-1} \subset I_{q,s-2} \subset \cdots \subset I_{1,0}$. Suppose $k$ corresponds to the adjacent to $I_{j,s}$ subinterval $I_{k,s}$ of $I_{m,s-1}$. Then $h_{m,s-1} \leq |z-t| \leq |y_j - y_k| \leq |z-t| + l_{j,s} + l_{k,s}$. Hence, $1 \leq \left| \frac{|y_j - y_k|}{|z-t|} \right| \leq 1 + \varepsilon_0$, where $\varepsilon_0 = \frac{l_{j,s} + l_{k,s}}{h_{m,s-1}} < \frac{1}{4}$, as $s \to \infty$.

For this $k$ we get

$$2^{-s} \log |y_j - y_k|^{-1} < \int_{I_{k,s}} \log |z-t|^{-1} d\lambda_s(t) < 2^{-s} \log |y_j - y_k|^{-1} + \varepsilon_0. \quad (10)$$

In its turn, $I_{q,s-2} \supset I_{m,s-1} \cup I_{n,s-1}$, where $I_{n,s-1}$ contains other two intervals of the $s$-th level. Let $k$ correspond to any of them. Then $|z-t| - l_{j,s} - l_{k,s} \leq |y_j - y_k| \leq |z-t| + l_{j,s} + l_{k,s}$ with $|z-t| \geq h_{q,s-2}$. Here, $1 - \varepsilon_1 \leq \left| \frac{|y_j - y_k|}{|z-t|} \right| \leq 1 + \varepsilon_1$ with

$$\varepsilon_1 = \frac{l_{j,s} + l_{k,s}}{h_{q,s-2}} < \frac{2}{7} \left( \frac{l_{j,s}}{h_{m,s-1}} + \frac{l_{k,s}}{h_{m,s-1}} \right) < \frac{8}{7} \cdot 4 \gamma_s \cdot 4 \gamma_{s-1} < \frac{1}{7} \cdot \frac{1}{4}, \quad (11)$$

by Lemma 3. Repeating this argument leads to the representation

$$\sum_{k=1, k \neq j}^{2^s} \int_{I_{k,s}} \log |z-t|^{-1} d\lambda_s(t) = 2^{-s} \log \prod_{k=1, k \neq j}^{2^s} |y_j - y_k|^{-1} + \varepsilon,$$

where $|\varepsilon| \leq 2^{-s+1}(\varepsilon_0 + 2 \varepsilon_1 + \cdots + 2^{s-1} \varepsilon_{s-1})$ with $\varepsilon_k < \frac{2}{7} \cdot 8^{-k}$ for $k \geq 1$. Here we used the estimate $\log(1 + x) \leq 2 \log |x|$ for $|x| < 1/2$. We see that $|\varepsilon| < 2^{-s}$.
The main term above is $2^{-s} \log |P_{2s}(y_j)|^{-1}$, which is $2^{-s} \log(\delta_s/r_s) + o(1)$, by Lemma 4. Thus,

$$\int \log |z - t|^{-1} d\lambda_s(t) = 2^{-s} \log(\delta_s/r_s) + o(1) \quad as \quad s \to \infty.$$ 

Finally, $2^{-s} \log(\delta_s/r_s) = R_s + 2^{-s} \log \frac{\delta_s}{r_s} \to R$ as $s \to \infty$, by (8). □

**Theorem 3.** Suppose $\gamma$ satisfies (5) and $\text{Cap}(K(\gamma)) > 0$. Then $\lambda_s \xrightarrow{\star} \mu_{K(\gamma)}$.

**Proof:** All measures $\lambda_s$ have unit mass. By Helly’s Selection Theorem (see for instance [15, Th.1.1.3]), we can select a subsequence $(\lambda_{s_k})_{k=1}^\infty$, weak* convergent to some measure $\mu$. Approximating the function $\log |z - \cdot|^{-1}$ by the truncated continuous function $\omega$ (see for instance [15, Th.1.6.9]), we get

$$\liminf_{k \to \infty} U^{\lambda_{s_k}}(z) = U^\mu(z)$$

for quasi-every $z \in \Sigma$. In particular, by Lemma 5, we have $U^\mu(z) = R$ for quasi-every $z \in K(\gamma)$. This means that $\mu = \mu_{K(\gamma)}$ (see e.g. [15, Th.1.3.3]). The same proof remains valid for any subsequence $(\lambda_{s_j})_{j=1}$. Therefore, $\lambda_s \xrightarrow{\ast} \mu_{K(\gamma)}$. □

**Remark.** Clearly, any compact set $K$ with nonempty interior cannot be equilibrium in our sense since $\text{supp} \mu_{K} \subset \partial K$. Neither geometrically symmetric Cantor-type sets of positive capacity are equilibrium. Let us consider the set $K^{(\alpha)}$ from [1] which is constructed by means of the Cantor procedure with $l_{s+1} = l_s^a$ for $1 < a < 2$. The values $\alpha \geq 2$ give polar sets $K^{(\alpha)}$. As above, let $\lambda_s$ be the normalized Lebesgue measure on $E_s = \bigcup_{j=1}^{2^s} I_{j,s}$. Given $s \in \mathbb{N}$, let $z_s = l_1 - l_2 + \cdots + (-1)^{s+1} l_s$. Estimating distances $|z - t|$ for $z = 0$ and $z = z_s$, as in Lemma 5, it can be checked that

$$U^{\lambda_s}(0) - U^{\lambda_s}(z_s) > \sum_{k=1}^{s-1} 2^{-k-1} \log \frac{(l_{k+1}-l_k)(l_{k+1}-l_{k+2})}{(l_{k-1}-l_k)(l_{k-1}-l_{k+2})}.$$ 

It is easily seen that all fractions here exceed 1. Therefore, for each $s$ there exists a point $z_s \in K^{(\alpha)}$ such that $U^{\lambda_s}(0) - U^{\lambda_s}(z_s)$ exceeds the constant $\frac{1}{4} \log \frac{(1-l_1)(1-l_2)}{(1-l_1)(1-l_2)}$ and the limit logarithmic potential is not equilibrium. Indeed, if $K^{(\alpha)}$ is not polar, then it is regular with respect to the Dirichlet problem (see [11]) and $U^\mu_{K^{(\alpha)}}$ must be continuous in $\Sigma$ and constant on $K^{(\alpha)}$.

6. Smoothness of $g_{\Sigma \setminus K(\gamma)}$

We proceed to evaluate the modulus of continuity of the Green function corresponding to the set $K(\gamma)$. Recall that a modulus of continuity is a continuous non-decreasing subadditive function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ with $\omega(0) = 0$. Given function $f$, its modulus of continuity is $\omega(f, \delta) = \sup_{|x - y| \leq \delta} |f(x) - f(y)|$.

In what follows the symbol $\sim$ denotes the strong equivalence: $a_s \sim b_s$ means that $a_s = b_s(1 + o(1))$ for $s \to \infty$. This gives a natural interpretation of the relation $\lesssim$.

Let $\gamma$ be as in the preceding theorem. Then, we are given two monotone sequences $(\delta_s)_{s=1}^\infty$ and $(\rho_s)_{s=1}^\infty$ where, as above, $\delta_s = \gamma_1 \cdots \gamma_s$, $\rho_s = \sum_{k=s+1}^{\infty} 2^{-k} \log \frac{1}{x_k}$. We define the function $\omega$ by the following conditions: $\omega(0) = 0$, $\omega(\delta) = \rho_1$ for $\delta \geq \delta_1$. If $s \geq 2$ then $\omega(\delta) = \rho_s + 2^{-s} \log \frac{\delta}{\delta_s}$ for $\delta_s \leq \delta \leq \delta_s/16$ and $\omega(\delta) = \rho_{s-1} - k_s(\delta_s - \delta) + 2^{-s} \delta_s^{-1} \log 8$ for $\delta_{s-1}/16 < \delta < \delta_{s-1}$ with $k_s = \frac{16}{15} \cdot 2^{-s} \delta_s^{-1} \log 8$.

**Lemma 6.** The function $\omega$ is a concave modulus of continuity. If $\gamma_s \to 0$ then for any positive constant $C$ we have $\omega(\delta) \sim \rho_s + 2^{-s} \log \frac{C\delta}{\delta_s}$ as $\delta \to 0$ with $\delta_s \leq \delta < \delta_{s-1}$.
Proof: The function $\omega$ is continuous due to the choice of $k_s$. In addition, $\omega'(\delta_{s-1} + 0) < k_s < \omega'(\delta_{s-1}/16 - 0)$, which provides concavity of $\omega$.

If $\gamma_s = \frac{1}{2} \exp[2s(\rho_s - \rho_{s-1})] \to 0$ then $2^s\rho_s \to \infty$ and we have the desired equivalence in the case $\delta_s \leq \delta \leq \delta_{s-1}/16$. Suppose $\delta_{s-1}/16 < \delta < \delta_{s-1}$. The identity

$$\rho_{s-1} = \rho_s + 2^{-s} \log \frac{\delta_{s-1}}{2\delta_s}$$

(10)
yields $|\rho_s + 2^{-s} \log \frac{C\delta}{\delta_s} - \omega(\delta)| < 2^{-s} [\log \frac{2C\delta}{\delta_{s-1}} + \frac{16}{15} \log 8 \cdot (1 - \frac{\delta}{\delta_{s-1}})] < 2^{-s} [\log C + 8 \log 2]$, which is $o(\omega)$ since here $\omega(\delta) > \rho_{s-1} - 2^{-s} \log \delta_s$.

**Lemma 7.** Suppose $\gamma$ satisfies (5) and $\text{Cap}(K(\gamma)) > 0$. Let $z \in \mathbb{C}$, $z_0 \in K(\gamma)$ with $\text{dist}(z, K(\gamma)) = |z - z_0| = \delta < 1$. Choose $s \in \mathbb{N}$ such that $z_0 \in I_{j,s} \subset I_{j_1,s-1}$ with $l_{j,s} \leq \delta < l_{j_1,s-1}$. Then $g_{\mathbb{C},K(\gamma)}(z) < \rho_s + 2^{-s} \log \frac{16\delta}{\delta_s}$.

On the other hand, if $l_{1,s} \leq \delta < l_{1,s-1}$ then $g_{\mathbb{C},K(\gamma)}(-\delta) > \rho_s + 2^{-s} \log \frac{\delta}{\delta_s}$.

Proof: Consider the chain of basic intervals containing $z_0$: $z_0 \in I_{j,s} \subset I_{j_1,s-1} \subset I_{j_2,s-2} \subset \cdots \subset I_{j_s,0} = [0, 1]$. Here, $I_{j,s,i} \setminus I_{j_1,s-1}$ contains $2^{-1}$ basic intervals of the $s$-th level. Each of them has certain endpoints $x, y$ with $x \in X_s, y \in Y_{s-1}$. Recall that $Y_{s-1}$ is the set of zeros of $P_{2s}$. Distinguish $y_j \in I_{js}$. Now for a fixed large $n$ we will express the value $|P_{2n}(z)| = \prod_{k=1}^{2n} |z - x_k|$ in terms of $\prod_{k=1,k \neq j}^{2^{s-1}} |y_j - y_k|$ (compare to Lemma 5). Clearly, each interval of the $s$-th level contains $2^{n-s}$ zeros of $P_{2^n}$, so we will replace these $2^{n-s}$ points with the corresponding $y_k$.

Let us first consider the product $\pi_0 := \prod_{x_k \in I_{j,s}} |z - x_k|$. Here, $|z - x_k| \leq \delta + l_{j,s} < 2\delta$, so $\pi_0 < (2\delta)^{2^{-s}}$.

Let $\pi_i := \prod_{x_k \in I_{m,s}} |z - x_k|$, where $I_{m,s}$ is adjacent to $I_{j,s}$. Then $|z_0 - x_k| \leq l_{j_1,s-1} = |y_j - y_m|$, since $y_j$ and $y_m$ are the endpoints of the interval $I_{j_1,s-1}$. Therefore, $|z - x_k| < 2|y_j - y_m| = \pi_1 < (2\delta)^{2^{-s}}$.

In the general case, given $2 \leq i \leq s$, let $\pi_i$ denote the product of all $|z - x_k|$ for $x_k \in J_i := I_{j_1,s-i} \setminus I_{j_1,s-i+1}$. Suppose $x_k \in I_{q,s}$. Then $|z - x_k| \leq \delta + l_{j,s} + |y_j - y_q| + l_{q,s} \leq |y_j - y_q|(1 + \frac{8\cdot l_{j_1,s-1}}{\delta})$, since $y_j$ and $y_q$ belong to different subintervals of the $(s-i+1)$-th level for $I_{j_1,s-i}$. Here, $\frac{\delta}{h_{j_1,s-1}} < \frac{8}{7} \frac{l_{j_1,s-1}}{l_{j_1,s-1}} < \frac{8}{7} \frac{8^{1-i}}{8^{1-i}}$, by Lemma 3. As in the proof of Lemma 5, we obtain $\frac{l_{j_1,s-1}}{h_{j_1,s-1}} < \frac{8}{7} \cdot 2 \cdot 8^{-i}$. From this, $\prod_{x_k \in I_{q,s}} |z - x_k| \leq [\frac{8}{7} \cdot 2 \cdot 8^{-i}]^{2^{n-s}}$. Since $J_i$ contains $2^{i-1}$ basic intervals of the $s$-th level, $\pi_i < [(1 + \frac{80}{7} \cdot 8^{-i})^{2^{i-1}} \prod_{y_k \in J_i} |y_j - y_q|]^{2^{n-s}}$.

The product $\prod_{i=0}^{s-1} [(1 + \frac{80}{7} \cdot 8^{-i})^{2^{i-1}}]$ is smaller than $2$, as is easy to check.

Therefore, $|P_{2^n}(z)| = \prod_{i=0}^{s-1} \pi_i < [8 \cdot \delta \cdot \prod_{k=1,k \neq j}^{2^{s-1}} |y_j - y_k|]^{2^{n-s}}$. The last product in the square brackets is $|P_{2^n}(y_j)|$, which does not exceed $r_s/\delta_s$, by Lemma 4. Hence, $2^{-n} \log |P_{2^n}(z)| < 2^{-s} \log \frac{16\delta}{\delta_s} - R_s$.

Finally, by Corollary 1, $g_{\mathbb{C},K(\gamma)}(z) = R + \lim_{n \to \infty} 2^{-n} \log |P_{2^n}(z)|$, which yields the desired upper bound of the Green function.

Similar, but simpler calculations establish the sharpness of the bound. We have $g_{\mathbb{C},K(\gamma)}(-\delta) = R + \lim_{n \to \infty} 2^{-n} \log P_{2^n}(-\delta)$. Now, $P_{2^n}(-\delta) = \prod_{i=0}^{s} \pi_i$ with $\pi_0 = \prod_{x_k \in I_{1,s}} (\delta + x_k) > \delta^{2^{n-s}}$ and $\pi_i = \prod_{x_k \in I_{2,s+i+1}} (\delta + x_k)$ for $i \geq 1$. Suppose $x_k \in I_{q,s} \subset \ldots \subset I_{j_{s-i},i} \subset I_{j_{s-i+1},i} \subset \ldots$
$I_{2,s-i+1}$. Then $\delta + x_k > y_q - l_{q,s}$. Since $y_q > h_{1,s-i} > \frac{q}{2} l_{1,s-i}$, we have $\delta + x_k > y_q(1 - \frac{q}{2} 8^{-i})$ and $\pi_i > [(1 - \frac{1}{4} 8^{-i})^{2-i} \prod_{q \in I_{2,s-i+1}} y_q]^2 n^{-s}$. Therefore, $P_{2^n}(-\delta) > \left[\frac{\delta}{2} \prod_{k=1}^{2^n} y_k\right]^{2^n} = \left[\frac{\delta}{2} |P_{2^n}(0)|\right]^{2^n} - \delta \delta_s \cdot r_s/2^{2n-s}$, by (2). Thus, $2^{-n} \log P_{2^n}(-\delta) > -R_s + 2^{-s} \log \frac{\delta}{\delta_s}$ and $g_{\gamma_s^\delta}(\delta) \geq \rho_s + 2^{-s} \log \frac{\delta}{\delta_s}$. □

**Theorem 4.** Suppose $\gamma$ satisfies (5) and $Cap(K(\gamma)) > 0$. If $\delta_s \leq \delta < \delta_{s-1}$ then $\rho_s + 2^{-s} \log \frac{\delta}{\delta_s} < \omega(g_{\gamma^s}(\gamma), \delta) < \rho_s + 2^{-s} \log \frac{16\delta}{\delta_s}$. If $\gamma_s \rightarrow 0$ then $\omega(g_{\gamma^s}(\gamma), \delta) \sim \omega(\delta)$ as $\delta \rightarrow 0$.

**Proof:** Fix $\delta$ and $s$ with $\delta_s < \delta < \delta_{s-1}$. By (7), $\delta_s < l_{1,s} < 2 \delta_s < \delta_{s-1}$.

If $l_{1,s} \leq \delta < \delta_{s-1}$ then $\omega(g_{\gamma^s}(\gamma), \delta) \geq g_{\gamma^s}(\gamma)(\delta)$, so Lemma 7 yields the desired lower bound. If $\delta_s \leq \delta < l_{1,s}$, then $g_{\gamma^s}(\gamma)(\delta) > \rho_{s+1} + 2^{-s-1} \log \frac{\delta}{\delta_s+1} = \rho_s + 2^{-s-1} \log \frac{2 \delta}{\delta_s}$, by (10). Here, $2^{-s-1} \log \frac{2 \delta}{\delta_s} > 2^{-s} \log \frac{2 \delta}{\delta_s}$, as is easy to check.

In order to get the upper bound, without loss of generality we can assume that $\omega(g_{\gamma^s}(\gamma), \delta) = g_{\gamma^s}(\gamma)(z)$ where $z \in \mathbb{C}$ is such that $dist(z, K(\gamma)) = |z - z_0| = \delta$ for some $z_0 \in K(\gamma)$.

Fix $m$ such that $z_0 \in I_{j,m} \subset I_{j+1,m-1}$ for some $j$ with $l_{j,m} \leq \delta < l_{j+1,m-1}$. Then $m \geq s$, since otherwise Lemma 4 gives a contradiction $\delta < \delta_{s-1} \leq \delta_m < l_{j,m} \leq \delta$.

If $m = s$ then, by Lemma 7, the result is immediate.

If $m \geq s+1$ then $g_{\gamma^s}(\gamma)(z) \leq \rho_m + 2^{-m} \log \frac{16\delta}{\delta_s}$ that does not exceed $\rho_s + 2^{-s} \log \frac{16\delta}{\delta_s}$. Indeed, the function $f(\delta) = \rho_m + 2^{-m} \log \frac{16\delta}{\delta_s}$ is such that for all $\delta_s \leq \delta < \delta_{s-1}$, $f(\delta_s) \leq 2^{-s-1} \log \frac{2 \delta}{\delta_s}$.

Instead of $\delta_s$, $\delta_{s+1}$ at the left endpoint. Here, $f(\delta_s) = (2^{-s} - 2^{-m}) \log 8 + \sum_{k=s+1}^{m}(2^{-k} - 2^{-m}) \log \frac{1}{\gamma_k} > 0$.

The last statement of the theorem is a corollary of Lemma 6. □

### 7. Model types of smoothness

Let us consider some model examples with different rates of decrease of $(\rho_s)_{s=1}^\infty$. Recall that for non-polar sets $K(\gamma)$ with $R = \operatorname{Rob}(K(\gamma))$ we have $\rho_s \downarrow 0$ and $R_s - R_{s-1} = \rho_{s-1} - \rho_s = 2^{-s} \log \frac{1}{\gamma_s}$ with $\rho_0 = R - \log 2$. Therefore, $R = \log 2 - \sum_{k=1}^\infty 2^{-k} \log 2\gamma_k$.

In addition, (5) implies $\rho_s \geq 2^{-s} \log 16$ and $R \geq 32$, so $Cap(K(\gamma)) \leq 1/32$.

If a set $K$ is uniformly perfect, then the function $g_{\gamma_s^s}$ is Hölder continuous (see e.g. [10], p. 119), which means the existence of constants $C, \alpha$ such that

$$g_{\gamma_s^s}(z) \leq C \cdot (\text{dist}(z, K))^\alpha$$

for all $z \in \mathbb{C}$.

In this case we write $g_{\gamma_s^s} \in \operatorname{Lip} \alpha$.

By Theorem 2, $g_{\gamma_s^s}$ is Hölder continuous provided $\gamma_s = \text{const.}$ Now we can control the exponent $\alpha$ in the definition above. In the following examples we suppose that $\text{dist}(z, K(\gamma)) = \delta$ with $\delta_s \leq \delta < \delta_{s-1}$ for large $s$.

**Example 2.** Let $\gamma_s = \gamma_1 \leq \frac{1}{32}$ for all $s$. Then $\delta_s = \gamma_1^s, r_s = \gamma_1^{s-1}, R = \log \frac{1}{\gamma_1}$, and $\rho_s = 2^{-s} \log \frac{1}{\gamma_1}$. Here, $\rho_s + 2^{-s} \log \frac{\delta}{\delta_s} \geq \rho_s > 2^{-s} = \delta_s^\alpha$ with $\alpha = -\frac{\log 2}{\log \gamma_1}$. Since $\delta_s = \gamma_1 \delta_{s-1} > \gamma_1 \delta$, we have, by Theorem 4, $g_{\gamma_s^s}(\gamma, \delta) > \gamma_1^\alpha \delta^\alpha$. On the other hand, $\rho_s + 2^{-s} \log \frac{16\delta}{\delta_s} < \delta^\alpha \log \frac{8}{\gamma_1}$. 10
Suppose we are given $\alpha$ with $0 < \alpha \leq 1/5$. Then the value $\gamma_s = 2^{-1/\alpha}$ for all $s$ provides $g_{C \setminus K(\gamma)} \in Lip \alpha$ and $g_{C \setminus K(\gamma)} \notin Lip \beta$ for $\beta > \alpha$.

The next example is related to the function $h(\delta) = (\log \frac{1}{\delta})^{-1}$ that defines the logarithmic measure of sets. Let us write $g_{C \setminus K} \in Lip \alpha$ if for some constants $C$ we have

$$g_{C \setminus K}(z) \leq C h^\alpha(\text{dist}(z, K)) \quad \text{for all} \quad z \in \mathbb{C}.$$  

**Example 3.** Given $1/2 < \rho < 1$, let $\rho_s = \rho^s$ for $s \geq s_0$, where $\frac{\rho}{1-\rho} \log 16 < (2\rho)^{s_0}$. This condition provides $\gamma_s < 1/32$ for $s > s_0$. Suppose $\gamma_s = 1/32$ for $s \leq s_0$, so we can use Theorem 4. For large $s$ we have $\delta_s = C 2^{-s} \mu(2^s)^\alpha$ with $\mu = \exp(\frac{2\rho^2}{\mu - 1})$ and some constant $C$. Let us take $\alpha = \frac{\log(1/\rho)}{\log(2\rho)}$, so $(2\rho)^\alpha = 1/\rho$. Then $h^\alpha(\delta) \geq h^\alpha(\delta_s) \geq \varepsilon_0(2\rho)^{-s\alpha} = \varepsilon_0 \rho \cdot \rho_{s-1}$ for some $\varepsilon_0$. From this we conclude that $g_{C \setminus K(\gamma)} \in Lip \alpha$ for given $\alpha$. Evaluation $g_{C \setminus K(\gamma)}(-\delta_s)$ from below yields $g_{C \setminus K(\gamma)} \notin Lip \beta$ for $\beta > \alpha$. Now, given $\alpha > 0$, the value $\rho = 2^{-\frac{1}{1-\alpha}}$ provides the corresponding Green function of the exact class $Lip \alpha$ (compare this to [1], [8]).

**Example 4.** Let $\rho_s = 1/s$. Then $\gamma_s = \frac{1}{2} \exp(\frac{-2^s}{s-2}) < 1/32$ for $s \geq 8$. As above, all previous values of $\gamma_s$ are $1/32$. Here, $\delta_s = C 2^{-s} \exp[\frac{2^s}{s} - \sum_{k=1}^{s-1} \frac{2^k}{k}]$. Summation by parts (see e.g. [14], T.3.41) yields $\delta_s = C 2^{-s} \exp[-2^{s+1}(s^{-2} + o(s^{-2}))]$. From this, $\omega(g_{C \setminus K(\gamma)}, \delta) \sim \frac{1}{s} \sim \frac{\log 2}{\log \delta}$.

**Example 5.** Given $N \in \mathbb{N}$, let $F_N(t) = \log \log \cdots \log t$ be the $N$-th iteration of the logarithmic function. Let $\rho_s = (F_N(s))^{-1}$ for large enough $s$. Here, $\rho_k = \rho_0 \sim [k \cdot \log k \cdot F_2(k) \cdots F_{N-1}(k) \cdot F^2_N(k)]^{-1}$. Since $\delta_s = C 2^{-s} \exp[-\sum_{k=1}^{s} 2^k \rho_0]$, we have, as above, $s \sim \frac{\log \log 1/\delta_s}{\log 2}$. Thus, $\omega(g_{C \setminus K(\gamma)}, \delta) \sim [F_{N+2}(1/\delta)]^{-1}$.

We see that a more slow decrease of $(\rho_s)$ implies a less smooth $g_{C \setminus K(\gamma)}$ and conversely. If, in examples above, we take $\gamma_s = 1/32$ for $s < s_0$ with rather large $s_0$, then the set $K(\gamma)$ will have logarithmic capacity as closed to 1/32, as we wish.

**Problem.** Given modulus of continuity $\omega$, to find $(\gamma_s)_{s=1}^\infty$ such that $\omega(g_{C \setminus K(\gamma)}, \delta)$ coincides with $\omega$ at least on some null sequence.

8. Markov’s factors

Let $P_n$ denote the set of all holomorphic polynomials of degree at most $n$. For any infinite compact set $K \subset \mathbb{C}$ we consider the sequence of Markov’s factors $M_n(K) = \inf \{ M : |P'|_K \leq M |P|_K \text{ for all } P \in P_n \}$, $n \in \mathbb{N}$. We see that $M_n(K)$ is the norm of the operator of differentiation in the space $(P_n, |.|_K)$. In the case of non-polar $K$, the knowledge about smoothness of the Green function near the boundary of $K$ may help to estimate $M_n(K)$ from above. The application of the Cauchy formula for $P'$ and the Bernstein-Walsh inequality yields the estimate

$$M_n(K) \leq \inf_{\delta} \delta^{-1} \exp[n \cdot \omega(g_{C \setminus K}, \delta)]. \quad (11)$$
This approach gives an effective bound of $M_n(K)$ for the cases of temperate growth of $\omega(g_{\mathcal{C}\setminus K})$. For instance, the Hölder continuity of $g_{\mathcal{C}\setminus K}$ implies Markov’s property of the set $K$, which means that there are constants $C, m$ such that $M_n(K) \leq Cn^m$ for all $n$. In this case, the infimum $m(K)$ of all positive exponents $m$ in the inequality above is called the best Markov’s exponent of $K$.

**Lemma 8.** Suppose $\gamma$ satisfies (5) and $\text{Cap}(K(\gamma)) > 0$. Given fixed $s \in \mathbb{N}$, let $f(\delta) = \delta^{-1} \exp[2^s(\rho_k + 2^{-k} \log \frac{16\delta}{\delta_k})]$ for $\delta_k \leq \delta < \delta_{k-1}$ with $k \geq 2$. Then $\inf_{\delta < \delta_1} f(\delta) = f(\delta_s - 0) = 4\sqrt{2} \delta_s^{-1} \exp(2^s \rho_s)$.

**Proof:** Let us fix the interval $I_k = [\delta_k, \delta_{k-1})$. In view of the representation $f(\delta) = C_{s,k} \delta^{2s-k-1}$, the function $f$ increases for $k < s$, decreases for $k > s$, and is constant for $k = s$ on $I_k$. An easy computation shows that $f(\delta_{k+1}) < f(\delta_k)$ for $k \leq s - 1$ and $f(\delta_{k-1} - 0) < f(\delta_k - 0)$ for $k \geq s + 1$. Thus, it remains to compare $f(\delta_s - 0)$ and $f(\delta_s)$. Here, $f(\delta_s) = 16 \delta_s^{-1} \exp(2^s \rho_s)$ exceeds $f(\delta_s - 0) = \delta_s^{-1}(16/\gamma_{s+1})^{1/2} \exp(2^s \rho_{s+1}) = 4\sqrt{2} \delta_s^{-1} \exp(2^s \rho_s)$. □

**Example 6.** Let $\gamma_s = \gamma_1 \leq \frac{1}{32}$ for $s \in \mathbb{N}$. Then, by Lemma 8 and Example 2, $M_{2s}(K(\gamma)) \leq \sqrt{8} \cdot \delta_s^{-1} = \sqrt{8} \gamma_1^{-1} 2^{s/\alpha}$, where $\alpha$ is the same as in Example 2.

On the other hand, let $Q = P_{2s} + r_s/2$. Then $|Q|_{K(\gamma)} = r_s/2$ and $|Q(0)| = r_s/\delta_s$, so $M_{2s}(K(\gamma)) \geq 2 \delta_s^{-1} = 2 \cdot 2^{s/\alpha}$. Now, for each $n$ we choose $s$ with $2^s \leq n < 2^{s+1}$. Since the sequence of Markov’s factors increases, $c_n^{1/\alpha} \leq M_2(K(\gamma)) \leq M_n(K(\gamma)) \leq M_{2s+1}(K(\gamma)) \leq C n^{1/\alpha}$

with $c = 2^{1-1/\alpha}$, $C = \gamma_1^{-1} 2^{3/2+1/\alpha}$. Given $m \in [5, \infty)$, the value $\gamma_s = 2^{-m}$ for all $s$ provides the set $K(\gamma)$ with $m(K(\gamma)) = m = 1/\alpha$.

However, the estimate (11) may be rather rough for compact sets with less smooth moduli of continuity of the corresponding Green’s functions. For instance, in the case of $K(\gamma)$ with $\sum_{k=1}^{\infty} \gamma_k < \infty$ (then $2^s \rho_s \to \infty$) and $n = 2^s$, the exact value of the right side in (11) is $4\sqrt{2} \delta_s^{-1} \exp(2^s \rho_s)$, whereas $M_{2s}(K(\gamma)) \sim 2 \delta_s^{-1}$, which will be shown below by means of the Lagrange interpolation. It should be noted that the set $K(\gamma)$ may be polar here.

Let us interpolate $P \in \mathcal{P}_{2^s}$ at zeros $(x_k)_{k=1}^{2^s}$ of $P_{2^s}$ and at one extra point $l_{1,s}$. Then the fundamental Lagrange interpolating polynomials are $L_s(x) = -P_{2^s}(x)/r_s$ and $L_k(x) = \frac{P_{2^s}(x)}{(x-x_{l_{1,s}})P_{2^s}(x_{l_{1,s}})}L_{2^s}(x_k)$ for $k = 1, 2, \ldots, 2^s$. Let $\Delta_s$ denote $\sup_{x \in K(\gamma)} |L'_s(x)| + \sum_{k=1}^{2^s} |L'_k(x)|$. For convenience we enumerate $(x_k)_{k=1}^{2^s}$ in increasing way, so $x_k \in I_{k,s}$ for $1 \leq k \leq 2^s$.

**Lemma 9.** Suppose $\gamma$ satisfies (5) and $\sum_{k=1}^{\infty} \gamma_k < \infty$. Then $\Delta_s \sim 2 \delta_s^{-1}$.

**Proof:** We use the following representation:

$$
L'_k(x) = \frac{P'_{2^s}(x)}{(x - x_{l_{1,s}})P_{2^s}(x_{l_{1,s}})} + \frac{P_{2^s}(x)}{(x - x_{l_{1,s}})P_{2^s}(x_{l_{1,s}})} \sum_{j=1, j \neq k}^{2^s} \frac{1}{x - x_j} =: A_k + B_k. \tag{12}
$$
In particular, $L'_1(0) = -l_{1,s}^1 - \sum_{j=2}^{2^s} x_k^1$. By (2), $|L'_s(0)| = \delta_s^{-1}$, so $\Delta s > |L'_s(0)| + |L'_1(0)| > \delta_s^{-1} + l_{1,s}^1 > \delta_s^{-1}(1 + e^{-16\gamma s})$, by (7). Thus, $\Delta s \gtrsim 2\delta_s^{-1}$.

We proceed to estimate $\Delta s$ from above. Lemma 4 gives the uniform bound $|L'_s(x)| \leq \delta_s^{-1}$.

Let us examine separately the sum $\sum_{k=1}^{2^s} |A_k|$, where $A_k$ are defined by (12). Let $C_0 = \exp(16 \sum_{k=1}^{\infty} \gamma_k)$. Then, by Lemma 4, $|P_{2^s}(x)| \leq |P_{2^s}(0)| = r_s/\delta_s < C_0 |P_{2^s}(x_k)|$ for $x \in K(\gamma)$. Therefore, $|A_1| \leq l_{1,s}^1 < \delta_s^{-1}$ and $\sum_{k=2}^{2^s} |A_k| < C_0 \sum_{k=2}^{2^s} (x_k - l_{1,s}^{-1})^{-1}$.

Here, $\sum_{k=2}^{2^s} (x_k - l_{1,s}^{-1})^{-1} < 2 l_{1,s-1}^{-1}$, as is easy to check. Thus, $\sum_{k=1}^{2^s} |A_k| < \delta_s^{-1} + 2C_0\delta_s^{-1}$.

In order to estimate the sum of the addends $B_K$, let us fix $x \in K(\gamma)$ and $1 \leq m \leq 2^s$ such that $x \in I_{m,s}$. Suppose first that $k \neq m$. Then

$$\sum_{j=1, j \neq k}^{2^s} \frac{P_{2^s}(x)}{x - x_j} < 2 \left| \frac{P_{2^s}(x)}{x - x_m} \right| \leq 2 \left| P_{2^s}(\xi) \right|$$

with a certain $\xi \in I_{m,s}$. Indeed, if $x = x_m$ then this sum is exactly $|P_{2^s}(x_m)|$, so $\xi = x_m$. Otherwise we take the main term out of the brackets:

$$\left| \frac{P_{2^s}(x)}{x - x_m} \right| \left[ 1 + \sum_{j=1, j \neq k, j \neq m}^{2^s} \left| \frac{x - x_m}{x - x_j} \right| \right].$$

Here the sum in the square brackets can be handled in the same way as in the proof of Lemma 3. Let $I_{m,s} \subset I_{q,s-1} \subset I_{r,s-2} \subset \cdots$. Then $[\cdots] \leq 1 + l_{1,m,s}(h_{q,s-1}^{-1} + 2h_{r,s-2}^{-1} + \cdots) \leq 1 + \frac{8}{7} l_{1,m,s}(1 + 2 + \cdots) < 1 + \frac{8}{7} (4\gamma s + 2 \cdot 4\gamma s 4\gamma s + \cdots) < 2.$

On the other hand, by Taylor’s formula, $P_{2^s}(x) = P_{2^s}(\xi)(x - x_m)$ with $\xi \in I_{m,s}$, which establishes (13).

Therefore,

$$\sum_{k=1, k \neq m}^{2^s} |B_K| < 2 \sum_{k=1, k \neq m}^{2^s} \left| \frac{P_{2^s}(x)}{x - x_k} \right|.$$
We are now in a position to construct a compact set with preassigned growth of subsequence of Markov’s factors. Suppose we are given a sequence of positive terms $(M_2^s)_{s=0}^\infty$ with $\sum_{s=0}^{\infty} M_2^s/M_2^{s+1} < \infty$. The case of polynomial growth of $(M_n)$ was considered before, so let us assume that $C n^m M_n^{-1} \to 0$ as $n \to \infty$ for fixed $C$ and $m$. Fix $s_0$ such that $M_{2^s}/M_{2^{s+1}} \leq 1/32$ for $s \geq s_0$ and $M_{2^{s_0}} \geq 2^{1/2}s_0$.

Let us take $\gamma_s = M_{2^{s-1}}/M_{2^s}$ for $s > s_0$ and $\gamma_s = (2/M_{2^{s_0}})^{1/s_0}$ for $s \leq s_0$. Then $\gamma_s \leq 1/32$ for all $s$ and we can use Theorem 5. Here, $\delta_s = 2/M_{2^s}$, so $M_{2^s}(K(\gamma)) \sim M_{2^s}$.

It should be noted that the growth of $(M_{n}(K))$ is restricted for a non-polar compact set $K$ ([5], Pr.3.1). It is also interesting to compare Theorem 5 with Theorem 2 in [16].

9. The best Markov’s exponent

If a compact set $K$ has Markov’s property, then the Markov inequality is not necessarily valid on $K$ with the best Markov’s exponent $m(K)$. An example of such compact set in $\mathbb{C}^N$, $N \geq 2$ was presented in [4], where the authors posed the problem (5.1): is the same true in $\mathbb{C}$? The compact set $K(\gamma)$ with a suitable choice of $\gamma$ gives the answer in the affirmative.

Example 7. Fix $m \geq 5$. Let $\varepsilon_k = \sqrt{k} - \sqrt{k-1}$ and $\gamma_k = 2^{-(m+\varepsilon_k)}$ for $k \in \mathbb{N}$. Then, $\delta_s = 2^{-(ms+\sqrt{s})}$ and $\rho_s = \sum_{k=s+1}^{\infty} 2^{-k} \log 2^{m-1+\varepsilon_k}$. Since $\varepsilon_k \leq 1$, we have $\exp(2^s \rho_s) < 2^m$. By Lemma 8 and (11), $M_{2^s}(K(\gamma)) < C_0 \delta_s^{-1}$ with $C_0 = 4\sqrt{2} \cdot 2^m$.

On the other hand, as in Example 6, $M_{2^s}(K(\gamma)) \geq 2^{s-1}$.

Let us show that for each $k \geq 2$ the value $m_k := m + \sqrt{k}/k-1$ is the Markov exponent for $K(\gamma)$. We want to find a constant $C_k$ such that $M_n(K(\gamma)) \leq C_k n^{m_k}$ holds for all $n \in \mathbb{N}$. Let $2^{s-1} < n \leq 2^s$. Then $M_n(K(\gamma)) \leq M_{2^s}(K(\gamma)) < C_0 2^m n^{m_s}$. If $s > k$ then $m_s < m_k$. If $s \leq k$ then $M_n \leq M_{2^k}$. Therefore, $C_k = \max\{C_0 2^m, M_{2^k}\}$ satisfies the desired condition.

However, the Markov inequality on $K(\gamma)$ does not hold with the exponent $m(K(\gamma)) = \inf m_k = m$. Indeed, $M_{2^s}(K(\gamma)) \geq 2^s \delta_s^{-1} = 2 \cdot 2^{m_s} \cdot 2^{\sqrt{s}}$. Therefore, given constant $C$, the inequality $M_{2^s}(K(\gamma)) \leq C 2^{m_s}$ is impossible for large $s$.

References


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