SMOOTHNESS OF THE GREEN FUNCTION FOR A SPECIAL DOMAIN.

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Dedicated to Professor J.Siciak on the occasion of his 80th birthday

Abstract. We consider a compact set $K$ in the form of a sequence of segments. By means of nearly Chebyshev polynomials for $K$, the modulus of continuity of the Green functions $g_{C\setminus K}$ is estimated. Markov’s constants of the corresponding set are evaluated.

1. Introduction

If a compact set $K$ is regular with respect to the Dirichlet problem then the Green function $g_{C\setminus K}$ of $\mathbb{C}\setminus K$ with pole at infinity is continuous throughout $\mathbb{C}$. We are interested in finding its modulus of continuity. The problem of smoothness of $g_{C\setminus K}$ near the boundary of $K$ was considered by many authors (see e.g. the references in the survey [3] and more recent [4],[10],[1]). A new impulse to investigate smoothness properties of the Green functions came in 2006, after appearance of the monograph [13] by V.Totik.

Here we consider a special compact set $K$ in the form of a sequence of segments. For the corresponding Green function we use the well-known representation

$$g_{C\setminus K}(z) = \sup \left\{ \frac{\log |P(z)|}{\deg P} : P \in \mathcal{P}, \deg P \geq 1, |P|_{K} \leq 1 \right\}. \quad (1)$$

Here and below, $|P|_{K}$ is the supremum norm of $P$ on $K$ and $\log$ denotes the natural logarithm. By $\mathcal{P}_n$ we denote the set of all complex polynomials of degree at most $n$, $\mathcal{P} = \bigcup_{n=0}^{\infty} \mathcal{P}_n$.

There are many sequences of polynomials that realize the supremum above, for example the normalized Fekete polynomials (see e.g. [9, T.11.1]), the normalized Chebyshev polynomials with zeros on $K$ (see e.g. [5, T.VII.4.4]), or any normalized sequence of polynomials orthogonal with respect to a regular (in the sense of [12]) measure. Following [7], we construct a sequence of “nearly Chebyshev” polynomials for $K$ and find the exact (up to a constant) value of the modulus of continuity of $g_{C\setminus K}$. It should be noted that the general bound by V.Totik ([13], T.2.2) of the Green functions, which is highly convenient to characterize optimal (that is Lip 1/2) smoothness of

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Let \( \phi \) be a nearly Chebyshev polynomial, that is \( \phi \) is such that for \( x \) near \( \delta \) the lower bound of \( g_{\mathbb{C} \setminus K}(\delta) \). The parameter \( \gamma \) here is such that \( g_{\mathbb{C} \setminus K}(\delta) \) is given in terms of the logarithmic measure (see e.g. [15]). The point \( \delta \) is regular, thus \( g_{\mathbb{C} \setminus K} \) is continuous throughout \( \mathbb{C} \).

We present the degrees \((n_k)_{k=1}^{m-1}\) such that the maximal values of \( |P_{N_m}(x)| \) for \( x \in I_k \) are smaller than \( 1 \) for \( 1 \leq k \leq m - 1 \). Clearly, \( |P_{N_m}(x)| < 1 \) for \( x \leq b_m \). We call \( P_{N_m} \) a nearly Chebyshev polynomial.

Substituting the polynomial \( P_{N_m} \) into (1) at \( z = -\delta \) yields the lower bound of \( g_{\mathbb{C} \setminus K}(-\delta) \).

In order to get the upper bound of \( g_{\mathbb{C} \setminus K}(\delta) \) such that the maximal values of \( |P_{N_m}(x)| \) for \( x \in I_k \) are smaller than \( 1 \) for \( 1 \leq k \leq m - 1 \). Clearly, \( |P_{N_m}(x)| < 1 \) for \( x \leq b_m \). We call \( P_{N_m} \) a nearly Chebyshev polynomial. The fundamental Lagrange polynomials are uniformly bounded by the desired value, which gives the main result.

Let \( \phi(\delta) = 1/ \log \frac{1}{\delta} \) for \( 0 < \delta < 1 \), \( \gamma = -\log(2-\sqrt{2})/ \log 2 \).

**Theorem 1.** Let \( \text{dist}(z, K) = \delta \leq b_1 \). Then \( g_{\mathbb{C} \setminus K}(\delta) \leq C \phi(\delta) \), where \( C \) does not depend on \( \delta \). On the other hand, \( g_{\mathbb{C} \setminus K}(-\delta) \geq \phi(\delta) \).

We remark that the modulus of continuity of \( g_{\mathbb{C} \setminus K} \) is given in terms of the function \( \phi \), which is used in the definition of the logarithmic measure (see e.g. [8, Ch.V, 6]). The parameter \( \gamma \) here is such that \( \phi(\delta) = (2-\sqrt{2})^\gamma \).

Given Green’s function \( g_{\mathbb{C} \setminus K} \), a standard application of the Cauchy formula for \( P' \) and the Bernstein-Walsh inequality allows to evaluate the Markov factors \( M_n(K) = \sup_{P \in P_n} |P'|_K / |P|_K \).

**Corollary 1.** There exists a constant \( C \) such that for \( n \in \mathbb{N} \) we have

\[
\exp n^{\gamma_1} \leq M_n(K) \leq \exp(C n^{\gamma_1}) \quad \text{with} \quad \gamma_1 = \frac{\log 2}{\log(2+\sqrt{2})}.
\]
3. Auxiliary results on the Chebyshev polynomials

Let $I = [a, b] = [c - \delta, c + \delta]$. We are interested in estimating the values of $T_{nl}(x) = T_n(\frac{x-c}{b})$ for $x \notin I$.

Let $\lambda_n(t) = (1 + \sqrt{1-t^2})^n + (1 - \sqrt{1-t^2})^n$ for $0 \leq t \leq 1$. Then, in view of the well-known representation $T_n(x) = 1/2 [(x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n]$ for $|x| \geq 1$ (see e.g. [11]), we get for $\Delta_1, \Delta_2 > \delta$

$$\frac{|T_{nl}(c + \Delta_1)|}{|T_{nl}(c + \Delta_2)|} = \left( \frac{\Delta_1}{\Delta_2} \right)^n \frac{\lambda_n(\delta/\Delta_1)}{\lambda_n(\delta/\Delta_2)}, \quad (2)$$

where the last fraction can be estimated in the following way.

**Lemma 1.** Let $0 < t_1 < t_2 < 1/2$. Then $\lambda_n(t_1)/\lambda_n(t_2) \leq 2 \exp[n(t_2^2 - t_1^2)/3]$.

**Proof:** We have $\lambda_n(t_1)/\lambda_n(t_2) < [(1 + \sqrt{1-t_1^2})^n + t_1^2]/(1 + \sqrt{1-t_2^2})^n = (1 + (t_2^2 - t_1^2)/a)^n + b^n$, where $a = (\sqrt{1-t_1^2} + \sqrt{1-t_2^2})(1 + \sqrt{1-t_2^2}) > 3$, $b = t_2^2/(1 + \sqrt{1-t_2^2}) < 1/7$, and the lemma follows. □

Next, we will use the corresponding monic polynomial $Q_{nl} = \delta^n 2^{1-n} T_{nl}$, that is $Q_{nl}(x) = \prod_{k=1}^n (x - x_k)$ with $x_k = c + \delta \cdot \xi_k$. Here, $\xi_k = \cos \frac{2k-1}{2n} \pi$ for $1 \leq k \leq n$ are the zeros of $T_n$. Since $T_n'(-\xi_k) = (-1)^{k-1} n / \sqrt{1 - \xi_k^2}$ (see e.g. [11], 1.24), and $n \leq |T_n'(-\xi_k)| \leq n / \sin(\pi/2n) < n^2$, we have

$$n \cdot (\delta/2)^n - 1 \leq |Q_{nl}(x_k)| < n^2 \cdot (\delta/2)^n - 1. \quad (3)$$

4. Nearly Chebyshev polynomials for $K$

The desired degrees $(n_k(m))_{m=1}^{m-1}$ will be defined by means of the sequence $(r_k)_{k=0}^\infty$, where $r_0 = r_1 = 1$ and $r_{k+1} = r_0 + r_1 + \cdots + r_{k-1} + 3 r_k$ for $k \geq 1$. This gives the second-order recurrence relation $r_{k+2} = 4 r_{k+1} - 2 r_k$, $k \geq 1$ with the solution $r_k = (2\sqrt{2})^{-1} (2 + \sqrt{2})^k - (2\sqrt{2})^{-1} (2 - \sqrt{2})^k$ for $k \geq 1$. We remark that all $r_k$ with $k \geq 2$ are even.

Given fixed $m \in \mathbb{N}$, we define $n_k = r_{m-k}$ for $k = 1, 2, \cdots, m$. Thus,

$$n_k = \frac{1}{2\sqrt{2}} \left[ (2 + \sqrt{2})^{m-k} - (2 - \sqrt{2})^{m-k} \right]$$

for $1 \leq k \leq m - 1$ and $n_m = 1$. If $m \geq 7$ then $n_m, n_{m-1}, \cdots, n_{m-6}, \cdots$ are given as $1, 1, 4, 14, 48, 164, 560, \cdots$. If $3 \leq q \leq m - 1$ then $n_q = 4 n_{q-1} - 2 n_q$. We collect together the properties of $(n_k)$ that will be used in what follows. Recall that $b_k = \exp(-2^k)$. The statements below follow on the definition of $(n_k)$ or straightforward calculations.

**Lemma 2.** Let $m \geq 3$. Then the numbers $n_k$ satisfy

1) $\sum_{k=q+1}^m n_k = n_{q-1} - 3 n_q$ for $2 \leq q \leq m - 1$
2) $(2\sqrt{2})^{-1} (2 + \sqrt{2})^{m-k} - 1 < n_k < (2\sqrt{2})^{-1} (2 + \sqrt{2})^{m-k}$ for $1 \leq k \leq m - 1$
3) $4 = n_{m-2}/n_{m-1} > n_{m-3}/n_{m-2} > \cdots > n_1/n_2 > 2 + \sqrt{2}$
4) $b_m \cdot b_{m-1} \cdot b_{m-2} \cdots b_{q+1} = b_q^{1+1+4+\cdots+q-n}$ for $1 \leq q \leq m - 1$
5) $N_m = \sum_{k=1}^m n_k = 1/2 \cdot [(2 + \sqrt{2})^{m-1} + (2 - \sqrt{2})^{m-1}] < (2 + \sqrt{2}) N_{m-1}$
6) \( n_k/N_m < (1+\sqrt{2}) 2^{-k} (2 - \sqrt{2})^k = (1+\sqrt{2}) (2+\sqrt{2})^{-k} \) for \( 1 \leq k \leq m-1 \)
7) \( \sum_{k=p}^{q} n_k < \sqrt{2} \cdot n_p \) for \( 1 \leq p \leq q \leq m-1 \)
8) \( \sum_{k=p}^{q} n_k / b_k < 2n_q/b_q \) for \( 1 \leq p \leq q \leq m \).

Suppose the polynomial \( P_{N_m} \) is defined by means of \( (n_k)_{k=1}^{m-1} \) and the compact set \( K \) is given as in Section 2.

**Lemma 3.** Given \( m \in \mathbb{N} \), let \( (n_k)_{k=1}^{m-1} \) be defined as above. Then \( |P_{N_m}(x)| \leq 1 \) for \( x \in K \).

**Proof:** The result is evident for \( P_1(x) = x/b_1 \) and \( P_2(x) = x/b_2(\epsilon_1-x)/\epsilon_1 \). Hence we can suppose that \( m \geq 3 \) and use Lemma 2.

Fix \( x \in K \). If \( x \leq b_m \) then \( 0 < T_{nk}(x)/T_{nk}(0) \leq 1 \) for all \( k \leq m-1 \), so \( |P_N(x)| \leq 1 \).

Suppose \( x \in I_q \) with \( 1 \leq q \leq m-2 \). Then \( 0 < T_{nk}(x)/T_{nk}(0) \leq 1 \) for \( 1 \leq k \leq q-1 \) and \( |T_{nq}(x)| \leq 1 \). Therefore, we need to check

\[
|b_q/b_m \cdot \prod_{k=q+1}^{m-1} |T_{nk}(b_q)/T_{nk}(0)| \leq |T_{nq}(0)|. \tag{4}
\]

From (2) we have \( |T_{nk}(b_q)/T_{nk}(0)| = (b_q-c_k/c_k)^{n_k} \lambda_{nk}(t_1) / \lambda_{nk}(t_2) \) with \( t_1 = b_{k+1}/2(b_q-c_k) < t_2 = b_{k+1}/2c_k \). Here, \( c_k = b_k - 1/2b_{k+1} \), so \( b_q-c_k/c_k < b_k/b_q \) and, by Lemma 1, \( \lambda_{nk}(t_1) < 2 \exp \frac{n_k t_1^2}{3} < 2 \exp(n_k b_k) \). Therefore the left side in (4) does not exceed

\[
\frac{b_{q+1}^{m-1} n_k \cdots n_{q+1}}{b_q \cdot b_{m-1} \cdot b_{m-2} \cdots b_{q+1}} 2^{m-1} \exp\left( \sum_{k=q+1}^{m-1} n_k b_k \right).
\]

Lemma 2 (4 and 8) now shows that the first fraction above is \( b_q^{-m} \) and

\[
\sum_{k=q+1}^{m-1} n_k b_k < 2n_{q+1} b_{q+1}, \text{ so the whole product is less than } b_q^{-m} 2^{m-1} e^{2n_{q+1} b_{q+1}}.
\]

On the other hand, the right side of (4) is \( \frac{1}{2} \left( \frac{c_q}{b_{q+1}/2} \right)^{n_q} \lambda_{nq}(b_{q+1}/2c_q) \). Clearly, \( \lambda_n(t) \geq 2 \) and \( 2c_q/b_{q+1} = 2b_q/b_q = 2b_q/b_q \). It follows that \( |T_{nq}(0)| \geq b_q^{-n_q} (2-b_q)^{n_q} \) and we only need to show that

\[
(m-q-1) \log 2 + 2n_{q+1} b_{q+1} < n_q \log(2-b_q).
\]

By Lemma 2 (3), it can be reduced to \( 2(m-q-1) \log 2 < n_q \), which is easy to check.

In the last case, when \( x \in I_{m-1} \), the condition (4) assumes the form

\[
b_{m-1}/b_m \leq |T_{1,m-1}(0)| = 2c_{m-1}/b_m,
\]

which evidently is fulfilled. \( \square \)

5. **Proof of the main result**

Let us prove at first the more simple sharpness result. Let us fix \( \delta \leq b_1 \) and \( s \geq 1 \) with \( b_{s+1} < \delta \leq b_s \). We consider the polynomial \( P = P_{N_{s+2}} \), that is \( P(x) = x/b_{s+2} \cdot \prod_{k=1}^{s+2} |T_{nk}(x)/T_{nk}(0)| \). By (1) and Lemma 3, we have
\[ g_{C \setminus K}(-\delta) \geq N^{-1} \log |P(-\delta)|. \]

Here, \( N = \frac{1}{2}(2 + \sqrt{2})^{s+1}[1 + (\sqrt{2} - 1)^{2s+2}] \), by Lemma 2 (5). Since \( |T_{n_k}(-\delta)| > |T_{n_k}(0)| \), we see that \( |P(-\delta)| > \delta/b_{s+2} > b_{s+1}^{-1} \) and

\[
g_{C \setminus K}(-\delta) > \frac{2^{s+1}}{N} = \frac{4}{(2 + \sqrt{2})[1 + (\sqrt{2} - 1)^{2s+2}]} \left( \frac{2}{2 + \sqrt{2}} \right)^s.
\]

The first fraction exceeds 1 and \((\frac{2}{2 + \sqrt{2}})^s = (2 - \sqrt{2})^s = \varphi^s(b_s)\), thus \( g_{C \setminus K}(-\delta) > \varphi^s(b_s) \geq \varphi^s(\delta) \), which is the desired conclusion.

We proceed to estimate \( g_{C \setminus K} \) from above. Let us first prove that

\[
g_{C \setminus K}(-b_s) \leq C(2 - \sqrt{2})^s \tag{5}
\]

for \( s \geq 2 \), where \( C \) does not depend on \( s \). In order to show this, we will define a certain increasing sequence \((D_m)\) with

\[
D_m < N_m < 2D_m, \tag{6}
\]

where \( N_m \) is given in Lemma 2(5). For each large \( m \) we will consider a system \((x_k)_{k=1}^{D_m}\) of interpolating points on \( K \). Then any polynomial \( P_N \) with \( D_m - 1 \leq N < D_m \) can be represented in the form \( P_N = \sum_{k=1}^{D_m} P_N(x_k) L_k \), where \((L_k)_{k=1}^{D_m}\) are the corresponding Lagrange fundamental polynomials.

We will show that for \( s \geq 2 \)

\[
\log |L_k(-b_s)| / N_m \leq C_1(2 - \sqrt{2})^s, \tag{7}
\]

where \( C_1 \) does not depend on \( s \) and \( k \).

Suppose \( |P_N|_K \leq 1 \). Then \( |P_N(-b_s)| \leq N_m \max_{1 \leq k \leq D_m} |L_k(-b_s)| \). By (6) and Lemma 2(5), we get \( N_m < 12(2 + \sqrt{2})N = C_2 N \). Therefore,

\[
\log |P_N(-b_s)| / N \leq \log(C_2 N) / N + C_1 C_2 (2 - \sqrt{2})^s.
\]

Since in the representation (1) we can consider only polynomials of arbitrary large degrees, the second term in the sum above dominates, which establishes the desired result (5).

We proceed to define the numbers \((D_m)\) and the corresponding interpolating points. Given \( s \geq 2 \), let us fix \( m \geq s + 2 \). Let \((n_q)_{q=1}^m\) and \( P_{N_m} \) be defined as above. The bound (7) is not valid if we use the zeros of \( P_{N_m} \) as interpolating points. For this reason we introduce new degrees \( d_q = n_q - \nu_q \) by means of the correction terms \( \nu_q = \lfloor n_q 2^{-q} \log 8 \rfloor \), where \( \lfloor x \rfloor \) denotes the greatest integer less than or equal to \( x \). We remark that \( d_q = n_q \) for large \( q \), namely \( q = m, m - 1, \ldots, m_1 \), with \( m_1 \approx m \cdot \frac{\log(2 + \sqrt{2})}{\log(4 + \sqrt{2})} \), whereas for small values of \( q \) the correction is essential. Since \( \nu_1 > n_1 \), we take \( d_1 = 0 \). An easy computation shows that

\[
8^{-n_q} \leq b_q^{\nu_q} < 8^{-n_q} b_q^{-1} \quad \text{for all} \quad q. \tag{8}
\]

Let us estimate the sum \( \sum_{k=q}^{m} \nu_k \) from above, where \( q \geq 2 \) and the actual summation is only till \( m_1 \), in view of the previous remark. By Lemma 2(2),

\[
\sum_{k=q}^{m} \nu_k \leq \frac{\log 8}{2\sqrt{2}} \cdot \sum_{k=q}^{m} \frac{(2 + \sqrt{2})^{m-k}}{2^k} < \frac{\log 8}{2\sqrt{2}} \cdot \frac{(2 + \sqrt{2})^{m-q}}{2^q} \cdot \sum_{k=0}^{\infty} (4 + 2\sqrt{2})^{-k}.
\]
We will denote the last sum by \( \rho \), so \( \rho = \frac{4+2\sqrt{2}}{3+2\sqrt{2}} \). On the other hand, the lower bound of \( n_k \) in Lemma 2(2) implies \( \nu_q > \frac{\log 8}{2^m} \left[ \frac{1}{2\sqrt{2}} (2 + \sqrt{2})^{m-q} - 1 \right] - 1 \). Thus,

\[
\sum_{k=q}^{m} \nu_k < \rho \cdot \nu_q + \rho \cdot (2^{-q} \log 8 + 1) < \rho \cdot \nu_q + 2, \tag{9}
\]

since \( \rho < \frac{6}{5} \) and \( q \geq 2 \).

Let us take \( x_1 = 0, x_2 = b_m \) and then \( d_q \) Chebyshev’s points on each interval \( I_q \) with \( q = m-1, m-2, \cdots, 2 \). That is, \( x_3 = c_{m-1} \), since \( d_{m-1} = n_{m-1} = 1 \) and \( T_{1,m-1}(x) = \frac{2(x-c_{m-1})}{b_m} \), then \( x_4, \cdots, x_7 \) are the zeros of \( T_{d_{m-2},m-2} \), etc. Thus, \( D_m := 1 + \sum_{q=2}^{m} d_q \) is the total number of interpolating points for given \( m \).

We proceed to show that (9) implies (6). Clearly, \( D_m = 1 + \sum_{q=2}^{m} (n_q - \nu_q) \) does not exceed \( N_m = \sum_{q=1}^{m} n_q \). The second inequality in (6) is equivalent to the following \( N_m < 12 (1 + N_m - n_2 - \sum_{k=2}^{m} \nu_k) \), which can be reduced, by (9), to \( 12 \rho \nu_2 + 12 + 12 n_1 < 11 N_m \). Here, \( \nu_2 \leq n_2 / 4 \cdot \log 8 \) and, by Lemma 2(6), \( N_m > \sqrt{2} n_1 \). Hence, it is enough to show that \( 3 \rho n_2 \log 8 + 12 < (11\sqrt{2} - 12) n_1 \). Since, by Lemma 2(3), \( n_1 > n_2 \cdot (2 + \sqrt{2}) \), we need to check that \( 12 < (10\sqrt{2} - 2 - 3 \rho \log 8) n_2 \). Recall that \( s \geq 2 \) and \( m \geq s + 2 \), so \( n_2 \geq n_{m-2} = 4 \). Finally, the inequality \( 5 + 3 \rho \log 8 < 10\sqrt{2} \) is valid for \( \rho < \frac{6}{5} \).

Our next goal is to prove (7). To shorten notation we write \( Q_j \) for the monic Chebyshev’s polynomial \( Q_{d_j,j} \) of degree \( d_j \) on \( I_j \), \( t_j := |Q_j(-b_s)|/Q_j(x_k) \).

Suppose first \( x_k \in I_q \) with \( s \leq q \leq m-1 \). Then \( |L_k(-b_s)| = \pi_1 \pi_2 \pi_3 \pi_4 \), where

\[
\pi_1 = \frac{b_s(b_s + b_m)}{x_k(x_k - b_m)} \prod_{j=q+1}^{m-1} t_j, \quad \pi_2 = \frac{|Q_q(-b_s)|}{(b_s + x_k)|Q_q(x_k)|}, \quad \pi_3 = \prod_{j=s}^{q-1} t_j, \quad \pi_4 = \prod_{j=2}^{s-1} t_j.
\]

For the terms in the product \( \pi_1 \) we have \( |Q_j(-b_s)| < b_s^{d_j} (1 + b_j/b_s)^{d_j} \) and \( |Q_j(x_k)| > b_s^{d_j} (1 - b_q - b_j/b_s)^{d_j} \). Therefore, \( \pi_1 < (b_s/b_q)^{1+d_m + \cdots + d_{q+1}} A_1 / B_1 \), where \( A_1 = \prod_{j=q+1}^{m} (1 + b_j/b_s)^{d_j} \leq \prod_{j=q+1}^{m} (1 + b_j/b_s)^{n_j} < e^{2n_q + b_q} \), by Lemma 2(8). Similarly, \( B_1 = (1 - b_q) \prod_{j=q+1}^{m} (1 - b_q - b_j/b_s)^{d_j} \). Since \( b_j/b_s \leq b_q \) and \( d_j \leq n_i \), we have, by Lemma 2(1), \( B_1 > (1 - 2b_q)^{1+n_m + \cdots + n_q} = (1 - 2b_q)^{1+n_q - 3n_q} > (1 - 2b_q)^{n_q - 1} \). Hence, \( B_1 > (1 + 3b_q)^{-n_q - 1} \). We can replace \( d_j \) by \( n_j \) also in the exponent of \( b_s/b_q \). Lemma 2(1) now yields \( \log \pi_1 < (n_q - 1 - 3n_q + 1)(2^q - 2^s) + b_q(2n_q + 1 + 3n_q) \). In addition, \( n_{q-1} \leq 4n_q \) and \( q \geq s \geq 2 \). Therefore, \( (n_{q-1} - 3n_q + 1)2^q \leq n_q 2^{q+1} \) and \( b_q(2n_q + 1 + 3n_q) < 4n_q \leq 2^s n_q \). Thus, \( \log \pi_1 < n_q 2^{q+1} \) and, by Lemma 2(6),

\[
\log \pi_1 / N_m < 2(1 + \sqrt{2})(2 - \sqrt{2})^q \leq (2 + 2\sqrt{2})(2 - \sqrt{2})^s.
\]
Let us estimate \( \pi_2 \) from above. The value \(|Q_q(-b_s)|\) consists of \( d_q \) terms. One of them coincides with \( b_s + x_k \). Hence, \( \frac{|Q_q(-b_s)|}{b_s + x_k} < (b_s + q)^{d_q-1} \leq (2b_s)^{d_q-1} \), as \( s \leq q \). On the other hand, by (3), \( |Q_q'(x_k)| > d_q (b_{q+1/4})^{d_q-1} \). Therefore,

\[
\pi_2 < (8b_s/b_{q+1})^{d_q-1} < b_q^{-d_q+1} < b_q^{-q} = \exp(n_q 2^{q+1}).
\]

By that, \( \log \pi_2/N_m \) has the desired bound.

Arguing as above, we see that

\[
\pi_3 < \frac{b_s^{q-1} \cdots b_s^{n_s} A_3}{b_s^{q-1} \cdots b_s^{n_s} B_3} \leq \frac{b_s^{q-1} \cdots b_s^{n_s} A_3}{b_s^{q-1} \cdots b_s^{n_s} B_3}
\]

with \( A_3 = \prod_{j=s+1}^{q-1} (1 + b_j/b_s)^{n_j} \), \( B_3 = \prod_{j=s+1}^{q-1} (1 - 2b_j)^{n_j} \). An easy computation shows that \( \log(A_3/B_3) \leq 2b_s(n_s+1 + 3n_s) \) and

\[
b_s^{q-1} \cdots b_s^{n_s} < B_3
\]

for \( s \geq 2 \). Thus, \( \pi_3 < (b_{q-1}^{q-1} \cdots b_{s+1}^{n_s})^{-1} \) and \( \log \pi_3 < \sum_{j=s+1}^{q-1} n_j 2^j \), so, by Lemma 2(6),

\[
\log \frac{\pi_3}{N_m} < (1 + \sqrt{2}) \sum_{j=s+1}^{q-1} (2 - \sqrt{2})^j < (2 + \sqrt{2})(2 - \sqrt{2})^s.
\]

To deal with \( \pi_4 \), we use (2):

\[
\pi_4 = \prod_{j=2}^{s-1} \left( \frac{c_j + b_s}{c_j - x_k} \right)^{d_j} \frac{\lambda_d(t_1)}{\lambda_d(t_2)}
\]

with \( t_1 = \frac{b_{j+1}}{2(c_j + b_j)} \), \( t_2 = \frac{b_{j+1}}{2(c_j - x_k)} \). Here, \( t_2 - t_1 < 2b_s b_j \). Hence, by Lemma 1 and Lemma 2(8),

\[
\prod_{j=2}^{s-1} \frac{\lambda_d(t_1)}{\lambda_d(t_2)} < 2^{s-2} \exp(2b_sn_2b_2).
\]

On the other hand, \( \frac{c_j + b_s}{c_j - x_k} < 1 + \frac{3b_s}{b_j} \). From this,

\[
\prod_{j=2}^{s-1} \left( \frac{c_j + b_s}{c_j - x_k} \right)^{d_j} < \exp(3b_s \sum_{j=2}^{s-1} n_j/b_j) < \exp(6n_{s-1}b_{s-1}).
\]

This and Lemma 2(6) imply that \( \log \pi_4/N_m < (s - 2) \log 2/N_m + 2b_s b_2(1 + \sqrt{2}) 2^{-2} (2 - \sqrt{2})^2 + 6b_s(1 + \sqrt{2})(2 + \sqrt{2})^{1-s}. \) Since \( N_m > (2 + \sqrt{2})^{m-1/2} \), the first term in the sum above does not exceed \( (2 - \sqrt{2})^s \). The same bound is valid for the second term, as \( 1 + \sqrt{2} < 2(2 - \sqrt{2}) s^{-2} e^4 e^{2s} \). For the last term we have \( 6b_{s-1}(1 + \sqrt{2})(2 + \sqrt{2})^{1-s} < 2(2 - \sqrt{2})^s \), since \( 3(4 + 3\sqrt{2}) < 2^s e^{2s-1} \) for \( s \geq 2 \). Therefore,

\[
\log \pi_4/N_m < 4(2 - \sqrt{2})^s,
\]

which is the desired conclusion.
Combining these we get (7) for \( x_k \in I_q \) with indicated values of \( q \). We note that above we did not use the difference between \( d_j \) and \( n_j \).

The cases \( k = 1 \) and \( k = 2 \) are simpler and very similar. For \( x_1 = 0 \) we have \[ |L_1(-b_s)| = \left( \frac{b_s + b_m}{b_m} \prod_{j=s}^{m-1} t_j \right) \left( \prod_{j=2}^{s-1} t_j \right) = \pi_3 \pi_4 \] with \( t_j := |Q_j(-b_s)| \).

We denote the corresponding parts of the product above by \( \pi_3 \) and \( \pi_4 \) because they are handled in the same way as \( \pi_3 \) and \( \pi_4 \) in the general case. Now,

\[ \pi_3 < \frac{b_{n_m + \ldots + n_{s+1}}}{b_{n_1 - 1} \ldots b_{n_{s+1}}^2 + 1} \frac{A_3}{B_3} \]

with \( A_3 = \prod_{j=s+1}^{m}(1 + b_j/b_s)^{n_j} \), \( B_3 = \prod_{j=s}^{m}(1 - b_j)^{n_j} \), so we can use the previous bound for \( A_3/B_3 \) and (10). Therefore, \( \pi_3 < (b_{n_m} \ldots b_{n_{s+1}})^{q_j} \) and the bound (11) is valid in this case as well.

Likewise, the value \( \pi_4 \) is the same as above if we take \( x_k = 0 \).

The same reasoning, with a minor modification of \( A_3 \) and \( B_3 \), applies to the case \( k = 2 \).

It remains to consider the most difficult case \( x_k \in I_q \) with \( 2 \leq q \leq s - 1 \). Recall that \( d_1 = 0 \), so the interval \( I_1 \) does not contain interpolating points.

Now we use the decomposition \[ |L_k(-b_s)| = \pi_1 \pi_2 \pi_3 \pi_4 \] with

\[ \pi_1 = \frac{b_s(b_s + b_m)}{x_k(x_k - b_m)} \prod_{j=s}^{m-1} t_j, \quad \pi_2 = \prod_{j=q+1}^{s-1} t_j, \quad \pi_3 = \frac{|Q_q(-b_s)|}{(b_s + x_k)|Q'_q(x_k)|}, \quad \pi_4 = \prod_{j=2}^{q-1} t_j, \]

where, as above, \( t_j \) means \( \frac{Q_j(-b_s)}{Q_q(x_k)} \). We note that \( \pi_4 = 1 \) for \( q = 2 \).

As before, \( \pi_1 < (b_s/b_q)^{1+d_m + \ldots + d_q} \cdot 2^{d_1} \cdot A_1/B_1 \) with \( A_1 = \prod_{j=s+1}^{m}(1 + b_j/b_s)^{d_j} < \exp(2 b_s n_{s+1}) \), \( B_1 = (1-b_q) \prod_{j=s}^{m}(1-b_q - b_j/b_q)^{d_j} > \exp(-3 b_q n_s) \).

Since, by Lemma 2(3), \( n_s \log 2 + 2 b_s n_{s+1} + 3 b_q n_s < n_s \), we get

\[ \pi_1 < (b_s/b_q)^{1+d_m + \ldots + d_q} \cdot e^{n_s}. \]

If \( q + 1 \leq j \leq s - 1 \), then \( b_j \) dominates \( b_s \). Therefore,

\[ \pi_2 < \frac{b_{d_{s-1}} \ldots b_{d_{q+1}}}{b_{d_{s-1} + \ldots + d_{q+1}}} \frac{A_2}{B_2} \]

with \( A_2 = \prod_{j=q+1}^{s-1}(1 + b_s/b_j)^{n_j} < \exp(2 b_{s-1} n_{s-1}) \), \( B_2 = \prod_{j=q+1}^{s-1}(1 - 2 b_q)^{n_j} \). 

Here, \( (1-2 b_q)^{-1} < 1+3 b_q \). Thus, \( B_2 > \exp(-3 b_q \sum_{j=q+1}^{s-1} n_j) > \exp(-3 \sqrt{2} b_q n_{q+1}) \), by Lemma 2(7). Combining these we get

\[ \pi_2 < \frac{b_{d_{s-1}} \ldots b_{d_{q+1}}}{b_{d_{s-1} + \ldots + d_{q+1}}} \cdot e^{2 b_{s-1} n_{s-1} + 3 \sqrt{2} b_q n_{q+1}}. \]

From (3) we obtain

\[ \pi_3 < \frac{(b_q + b_s)^{d_q - 1}}{d_q (b_{q+1}/4)^{d_q - 1}} = \frac{1}{d_q} \left( \frac{4}{b_q} \right)^{d_q - 1} \left( 1 + \frac{b_s}{b_q} \right)^{d_q - 1} < \frac{1}{d_q} \left( \frac{4}{b_q} \right)^{d_q - 1} e^{n_s b_q}. \]
Similarly to the case \( s \leq q \), for \( \pi_4 \) we use (2): 
\[
\pi_4 = \prod_{j=2}^{q-1} \left( \frac{\nu_j + b_j}{\lambda_j(t_j)} \right)^{d_j} \] 
with \( t_1 = \frac{\nu_{q-1}}{\lambda_{q-1}(t_{q-1})} \), \( t_2 = \frac{\nu_{q-1}}{\lambda_{q-1}(t_{q-1})} \). As above, \( t_2^2 - t_1^2 = \frac{\nu_{q-1}^2 (\nu_{q-1} + b_{q-1})}{\lambda_{q-1}(t_{q-1})^2} \). 

Here, \( b_s + x_k < 2b_q \), \( 2c_j + b_s - x_k < 2c_j + b_s \), and \( (c_j + b_s)(c_j - x_k) > b_j^2 / 2 \).

Hence, \( t_2^2 - t_1^2 < 2b_q b_j \). By Lemma 1 and Lemma 2 (8), \( \prod_{j=2}^{q-1} \frac{\nu_j - \lambda_j(t_j)}{\lambda_{q-1}(t_{q-1})} < 2^q \exp(\frac{4}{9} b_q n_2 b_2) \). Also, \( \frac{\nu_j + b_j}{\lambda_j(t_j)} < 1 + \frac{b_j}{b_j} \) and \( \prod_{j=2}^{q-1} \frac{\nu_j + b_j}{\lambda_j(t_j)} < \exp(4 n_q - b_q) \).

Therefore, 
\[
\pi_4 < \exp(4 n_q - b_q - 1 + q + 4/3 \cdot b_q n_2 b_2) \quad \text{for} \quad q \geq 3.
\]

Combining all inequalities yields 
\[
|L_k(-b_s)| < \frac{b^{1+n \nu_1 + \cdots + n \nu_{q-1} + \nu_{q-1} + \cdots + \nu_q + n \nu_1 + \cdots + n \nu_{q-1}}}{b_q \cdot b_q \cdot \ldots \cdot b_q} \cdot e^{\nu_1 + \nu_2},
\]
where \( \nu_1 = n_s + 2b_s - n_s + n_s b_q + q < 2n_s + q \), \( \nu_2 = 3\sqrt{2} b_q n_q + 4 n_q - 1 + q + 4/3 \cdot b_q n_2 b_2 \) and \( \nu_1 + \nu_2 + \cdots \) denotes a sum of all nonzero correction terms starting from \( \nu_i \).

Let us consider the first fraction in the product above. By Lemma 2(4), \( b_1 k_{s+n}^s = \prod_{k=q+1}^{k=n} b_{k^n} \). Therefore the numerator of this fraction equals \( b_{1-n}^q n_{1-n}^k \). On the other hand, its denominator is just \( \prod_{k=q+1}^{k-n} b_{k^n} \), by the same reason. Hence the first fraction above is \( b_{s-n}^q \).

Further, \( b_{q-1}^{\nu_q + \nu_{q+1} + \cdots} \leq b_{q-1}^{\nu_q + \nu_{q+1}} = b_{q-1}^{\nu_q + \nu_{q+1} + 1/2} < 8b_q n_q \cdot b_{q-1}^{\nu_q + \nu_{q+1}} \), by (8).

In turn, by (9), the denominator of the second fraction is larger than \( b_{s-2}^{\nu_1 + \nu_2} \cdot 2n_q \cdot b_{s-2}^{\nu_1 + \nu_2} \). Thus the second fraction does not exceed \( b_{s-2} b_{s-2}^2 \), by (8), where \( \kappa = n_q + 1/2 n_q - 1 - n_q + 1 + \cdots + n_q + 1 + \rho \cdot n_q \). Here, \( \rho \cdot n_q < n_q + n_s + n_{q+1} \).

Hence, by Lemma 2 (7), \( \kappa > n_q - (\sqrt{2} - 1/2) n_q \). Therefore, 
\[
|L_k(-b_s)| < b_{s-2-n}^q \cdot b_{q-2} \cdot e^{n_q + q} \cdot \frac{4 n_q}{\kappa} \cdot e^{\nu_q}.
\]

From a computational point of view, we introduce correction terms \( (\nu_q) \) in order to neutralize \( 4^q \) in the numerator above, which is unacceptably large if we use the degrees \( n_q \) without correction.

According to the estimation of \( \kappa \), we have \( 4^{n_q} < 8^q \). Let us show that \( e^{\nu_2} \leq 4^{\nu_2+1} \).

If \( q = 2 \) then \( \pi_4 = 1 \) and \( \mu_2 \) contains only its first term, that is \( \mu_2 = 3\sqrt{2} b_q n_3 \). By Lemma 2(3), \( \mu_2 < \frac{3\sqrt{2}}{2 + \sqrt{2}} e^{-4} n_2 \). On the other hand, \( (\nu_2 + 1) \log 4 > \frac{n_q}{q_2} \cdot 6 \log^2 2 \), which exceeds \( \mu_2 \).

Thus we can suppose that \( q \geq 3 \). Since \( q \leq s-1 \) and \( m \geq s+2 \), we have \( n_q = n_q - n_q < \frac{q}{2} \), by Lemma 2 (3). Also, \( n_q < 2(2 + \sqrt{2}) q^2 n_q \), by Lemma 2 (2). Therefore, \( \mu_2 < n_q (\frac{3\sqrt{2}}{2 + \sqrt{2}} b_q + 14 b_q - 1 + \frac{8}{3} (2 + \sqrt{2}) q^2 b_q b_2) \). On the other hand, \( (\nu_2 + 1) \log 4 > \frac{n_q}{2q} \cdot 6 \log^2 2 \). It is enough to show that...
\[
2^q \left[ \frac{3 \sqrt{2}}{2 + \sqrt{2}} e^{-2^q} + 14 e^{-2^q - 1} + \frac{8}{3} (2 + \sqrt{2})^{q-2} e^{-2^q} e^{-4} \right] < 6 \log^2 2.
\]

The expression on the left side attains its maximal value at the minimal \( q \), so we reduced the proof to the case \( q = 3 \), which can be checked by a straightforward calculation.

From here,

\[
\log |L_k(-b_s)| < (n_s + 1) 2^s + 2^{q+1} + 2 n_s + q + 2 \leq n_s (2^s + 2) + 2^{q+1} + s + 1,
\]

since \( q + 1 \leq s \). Recall that \( m \geq s + 2 \), so \( n_s \geq n_{m-2} = 4 \). Therefore, \( \log |L_k(-b_s)| < n_s 2^{s+2} \), which gives (7) in view of Lemma 2(6). This completes the proof of (5).

We now turn to the general case. Suppose \( \text{dist}(z, K) = \delta \leq b_1 \). We want to show that \( g_{C \setminus K}(z) \leq C \varphi^\gamma(\delta) \), where \( C \) does not depend on \( \delta \). We can assume, by increasing \( C \) if necessary, that \( \delta \leq b_{s_0} \) for any \( s_0 \) given beforehand. Let us take \( s_0 = 4 \). Fix \( z \) with \( \text{dist}(z, K) = \delta \leq b_4 \) and \( s \geq 4 \) such that \( b_{s+1} < \delta \leq b_s \). Since \( \frac{1}{2} \varphi(b_s) = \varphi(b_{s+1}) < \varphi(\delta) \leq \varphi(b_s) \), it is enough to show that

\[
g_{C \setminus K}(z) \leq C (2 - \sqrt{2})^s
\]

(12)

for \( z \) with \( \text{dist}(z, K) = b_s \), where \( C \) does not depend on \( s \).

Suppose first that \( \text{dist}(z, K) = |z - z_0| \) with \( z_0 \in I_q \) for some \( q \) with \( q \leq s - 2 \). The monotonicity of the Green function with respect to the set \( K \) implies that \( g_{C \setminus K}(z) \leq g_{C \setminus I_q}(z) \). It is well-known that, given \( I = [-l, l] \), the Green function \( g_{C \setminus I}(z) = \log |\frac{1}{2} + \sqrt{(\frac{1}{2})^2 - 1}| \) attains its maximal value, among all \( z \) with \( \text{dist}(z, I) = \delta \), at the real points. Therefore,

\[
\max \{ g_{C \setminus I}(z) : \text{dist}(z, I) = \delta \} = g_{C \setminus I}(l + \delta) \leq 2 \sqrt{\delta/\ell}
\]

if \( \delta \leq l/4 \). In our case, \( g_{C \setminus I_q}(z) \leq 2 \sqrt{2b_{s}/q_{q+1}} = 2 \sqrt{2} \exp(2^q - 2^{q-2}) \). Since \( q \leq s - 2 \), we have \( g_{C \setminus K}(z) \leq 2 \sqrt{2} \exp(-2^{s-2}) \), which does not exceed \((2 - \sqrt{2})^s \) for \( s \geq 4 \). This gives (12) for the first case.

It remains to consider \( z_0 \in K \cap [0, b_{s-1}] \). Recall that in the main bound (7) we estimated Lagrange fundamental polynomials with interpolating points \( (x_k)_{k=1}^{D_n} \). Let us compare distances from these points to \( z \) and, respectively, to the point \( -b_{s-2} \). If \( x_j \leq b_{s-1} \) then \( |z - x_j| \leq |z - z_0| + |z_0 - x_j| \leq b_s + b_{s-1} < b_{s-2} < b_{s-2} + x_j \). Otherwise, \( x_j \geq a_{s-2} \) and \( |z - x_j| \leq |z - z_0| + x_j = b_s + x_j < b_{s-2} + x_j \).

It follows that \( |L_k(z)| = \prod_{j=1,j \neq k}^{D_n} \frac{|z-x_j|}{|x_k-x_j|} < |L_k(-b_{s-2})| \). Here, \( s - 2 \geq 2 \), so we can apply (7). Arguing as above, we can generalize (5) to (12).
6. On Totik’s bound

In 2006 V.Totik ([13], T.2.2) obtained the following remarkable estimate of the Green function (we formulate it for a compact set $K \subset [0,1]$)

$$g_{C \setminus K}(-\delta) \leq C \sqrt{\delta} \exp \left( D \int_{\delta}^{1} \frac{\Theta_{K}^{2}(t)}{t^{3}} dt \right) \log \frac{2}{\text{cap}(K)}.$$  \hfill (13)

Here, $C, D$ are absolute constants, it is supposed that $K$ is not polar, and the function $\Theta_{K}$ is defined in our case as $\Theta_{K}(t) = m([0, t] \setminus K)$, where $m$ stands for the linear Lebesgue measure. In the case $I_{\Theta_{K}} := \int_{0}^{1} \Theta_{K}^{2}(t) t^{-3} dt < \infty$, the Green function $g_{C \setminus K}$ has $\text{Lip}^{1}$ smoothness, which is optimal for compact sets on $\mathbb{R}$. V.Totik proved that the condition of convergence of the integral is sharp: given function $\Theta$ with $I_{\Theta} = \infty$ there exists a set $K$ with $\Theta_{K} \leq \Theta$ whose Green function is not from $\text{Lip}^{1}$ class at the origin.

Thus, the estimation above is very appropriate to analyze boundary behavior of the Green functions with optimal smoothness. However, for compact sets with divergent $\text{cap}$ sets with divergent $I_{\Theta_{K}}$, the general estimate maybe rough, because of uncontrollable constant $D$. For example in our case, $\int_{b_{n}}^{b_{n+1}} \Theta_{K}^{2}(t) t^{-3} dt > 2^{s} - 2$, so the right side of (13) exceeds $Cb_{s}^{D+1/2}$ for $\delta = b_{s}$.

Neither can the previous general bound of the Green functions by M.Tsuji [14, Th.III.67], be applied for the compact set considered in the paper. In fact, (13) is the refinement of the estimate by M.Tsuji.

It is also interesting to apply the lower bound by V.Andrievskii ([2] or T.2.3 in [3]) to our case. We get $g_{C \setminus K}(-b_{s}) > \frac{1}{10} b_{s}^{1/2-\epsilon}$ with rather small $\epsilon$.

7. Markov’s factors

Let us show that for $\gamma_{1} = \frac{\log 2}{\log(2+\sqrt{2})}$ and some constant $C$ we have

$$\exp n^{\gamma_{1}} \leq M_{n}(K) \leq \exp(Cn^{\gamma_{1}}) \text{ for } n \in \mathbb{N}.$$  \hfill (13)

Suppose that for some increasing continuous function $F$ we have the bound $g_{C \setminus K}(z) \leq F(\delta)$ for $\text{dist}(z, K) \leq \delta$. The application of the Cauchy formula for $P^{\prime}$ and the Bernstein-Walsh inequality gives (see e.g. [1]) the estimate

$$M_{n}(K) \leq \inf_{\delta} \delta^{-1} \exp[n \cdot F(\delta)].$$  \hfill (13)

In our case $F(\delta) = C \varphi(\delta)$ and the value $\delta$ with $(\log \frac{1}{\delta})^{1+\gamma} = C n$ gives the desired upper bound of $M_{n}(K)$.

On the other hand, let us fix $n \in \mathbb{N}$ and $m$ with $N_{m} \leq n < N_{m+1}$, where $N_{m}$ is given in Lemma 2(5). For the polynomial $P = P_{N_{m}}$ from Section 2 we have $|P'(0)| = b_{m}^{-1} = \exp 2^{m}$ and $|P|_{K} \leq 1$, by Lemma 3.

Since the sequence $(M_{n}(K))$ is not decreasing, we get $M_{n}(K) \geq M_{N_{m}}(K) \geq |P'(0)| / |P|_{K} \geq \exp 2^{m}$. The last value exceeds $\exp N_{m+1}^{\gamma_{1}}$, since $N_{m+1} < (2 + \sqrt{2})^{m}$. This completes the proof of Corollary as $N_{m+1} > n$.  \hfill (13)
References


