On characterization of the extension property

A.P. GONCHAROV

Department of Mathematics, Bilkent University,
Ankara, 06800, Turkey
* E-mail: goncha@fen.bilkent.edu.tr

The geometric characterization of the extension property for Cantor-type sets, found in [3], is related to the rate of growth of the values of the discrete logarithmic energies of compact sets that locally form the set.

Keywords: Extension property, Cantor-type sets, discrete logarithmic energies.

1. Extension Problem

Let $K$ be a compact set in $\mathbb{R}^d$. Then $\mathcal{E}(K)$ is the space of Whitney jets on $K$, that is the space of traces on $K$ of $C^\infty$ functions. Topology in the space $\mathcal{E}(K)$ can be given by the system of seminorms

$$||f||_q = \inf |F|_q, \quad q \in \mathbb{N},$$

where the infimum is taken for all possible extensions of $f$ to $F$ and $|F|_q$ denotes the $q$–th norm of $F$ in $C^\infty(\mathbb{R}^d)$.

The Extension Problem is to characterize when there exists a linear continuous extension operator $L : \mathcal{E}(K) \rightarrow C^\infty(\mathbb{R}^d)$. We say that the compact set $K$ has the extension property if there exists a such operator.

Tidten in [6] applied Vogt’s condition for a splitting of exact sequences of Fréchet spaces and gave a topological characterization of the extension property: a compact set $K$ has the extension property if and only if the space $\mathcal{E}(K)$ has a dominating norm (see for instance [2] for the definition of a dominating norm and for a recent account of the theory).

Nevertheless, the problem of a geometric characterization of the extension property that goes back to the work [5] of Mityagin, is still open even for the one-dimensional case, in spite of the presence of numerous particular results. Here we consider the geometric characterization of the extension property for Cantor-type sets found in [3].
2. Cantor-type sets

Given \( l_1 \) with \( 0 < l_1 < 1/2 \) and a sequence \((\alpha_s)_{s=0}^{\infty}\) with \( \alpha_s > 1 \) let us define the sequence \((l_s)_{s=0}^{\infty}\) in the following way: \( l_0 = 1, l_1, l_2 = l_1^{\alpha_2}, \ldots, l_s = l_1^{\alpha_2 \cdots \alpha_s}, \ldots \). Then by \( K(\alpha_s) \) we denote the symmetric Cantor-type set \( \bigcap_{s=0}^{\infty} \bigcup_{j=1}^{2^s} I_{j,s} \), where \( |I_{j,s}| = l_s \) for all \( j \). Here the closed intervals \( I_{j,s} \) we call basic intervals. Let \( x \) be an endpoint of some basic interval. Then there exists the minimal number \( q \) (the type of \( x \)) such that \( x \) is the endpoint of some \( I_{j,m} \) for every \( m \geq q \).

As in [3] we suppose that \( \alpha_s \geq 1 + \varepsilon_0, s \geq s_0 \) for some positive \( \varepsilon_0 \) and \( l_s \geq 4 l_{s+1} \) for all \( s \). Let \( h_s = l_s - 2l_{s+1} \) be the gap between two adjacent intervals.

We follow the notations used in [3]: \( \pi_{n,0} = 1 \) and for \( n \geq 1, s \geq 1 \) let
\[
\pi_{n,s} = 2^{-s} \alpha_{n+1} \alpha_{n+2} \cdots \alpha_{n+s}, \quad \sigma_{n,s} = \sum_{k=0}^{s} \pi_{n,k}.
\]

**Theorem** [3]. The following are equivalent:

(i) The set \( K(\alpha_n) \) has the extension property.

(ii) \( \forall M > 0 \exists s_M : \sum_{n=1}^{M} \pi_{n+s} > \sum_{n=1}^{M} \pi_{n+s} \cdot 4 \sum_{n=1}^{M} \pi_{n+s}, \quad \forall n \quad \forall s \geq s_M \).

(iii) \( \sigma_{n,s+1} / \sigma_{n,s} \rightarrow 1 \), as \( s \rightarrow \infty \) uniformly with respect to \( n \).

We see that the condition (ii) is purely geometrical, whereas the condition (iii) is related to the theory of logarithmic potential. In what follows \( \log \) denotes the natural logarithm.

3. Discrete logarithmic energies

Let \( K \) be a compact set in \( \mathbb{C} \) and for given \( N \) points \( z_1, \ldots, z_N \subset K \) let \( \mu_N = \mu_N(z_1, \ldots, z_N) \) denote the discrete measure that associates the mass \( 1/N \) to any point \( z_k, 1 \leq k \leq N \). The logarithmic potential of measure \( \mu_N \) is given by
\[
U^{\mu_N}(z) = \frac{1}{N} \sum_{k=1}^{N} \log \frac{1}{|z - z_k|}.
\]

Any discrete measure has infinite energy, but if we use the truncated kernels (see e.g. [1]), then we can define the corresponding logarithmic energy as in [4]:
\[
I(\mu_N) = \frac{1}{N^2} \sum_{k,j=1,k \neq j}^{N} \log \frac{1}{|z_j - z_k|}.
\]
Clearly,

\[ I(\mu_N) = \frac{-2}{N^2} \log |V(z_1, \ldots, z_N)|, \]

where for the corresponding Vandermonde determinant we have

\[ V(z_1, \ldots, z_N) = \prod_{1 \leq j < k \leq N} (z_k - z_j). \]

Points \((z_k)_{k=1}^N \subseteq K\) that maximize the determinant \(V(z_1, \ldots, z_N)\) (or minimize the corresponding discrete logarithmic energy) are known as Fekete points of order \(s\) for \(K\).

Given compact set \(K^{(s_n)}\), let us fix \(n \in \mathbb{N}\) and \(N = 2^{s+1}\) for some \(s \in \mathbb{N}_0 : = \{0, 1, \ldots \}\).

Let \((z_k)_{k=1}^N\) consist of all endpoints of the type \(\leq s + n\) on the first basic interval \([0, l_n]\) ordered increasingly, that is \(z_1 = 0, z_2 = l_{s+n}, z_3 = l_{s+n+1} - l_{s+n}, \ldots, z_N = l_n\).

**Lemma 3.1.**

\[ e^{-2^{s+3} \prod_{k=0}^s \sigma_{n,s}} < |V(z_1, \ldots, z_N)| < \prod_{k=1}^s \sigma_{n,s}. \]

**Proof.** Temporarily we denote \(l_{n+s} l_{n+s-1}^2 \cdots l_n^s\) by \(\lambda\). We see at once that \(\prod_{k=2}^N |z_k - z_1| < \lambda\). The product \(\prod_{k=3}^N |z_k - z_2|\) with the upper bound \(l_{n+s-1}^2 \cdots l_n^s\) we join with \(|z_N - z_{N-1}| = l_{n+s}\). We continue in this fashion to join the product \(\prod_{k=m+1}^N |z_k - z_m|\) with the product \(\prod_{k=1}^{N-m+2} |z_k - z_{N-m+1}|\) for \(m = 3, \ldots, 2^s\). For each \(m\) we can estimate from above the joint product through \(\lambda\). Since there are \(2^s\) pairs of products, we get the bound

\[ |V(z_1, \ldots, z_N)| < \lambda^{2^s}. \]

By definition, \(\lambda\) has the form \(l_n^s\), where \(\lambda = \alpha_{n+1} \alpha_{n+2} \cdots \alpha_{n+s} + 2\alpha_{n+1} \cdots \alpha_{n+s-1} + \ldots + 2^s - 1 \alpha_{n+1} + 2^s = 2^s [1 + 2^{-1} \alpha_{n+1} + \cdots + 2^{-s} \alpha_{n+1} \alpha_{n+2} \cdots \alpha_{n+s}] = 2^s \sigma_{n,s},\) which gives the desired upper bound of the lemma.

To estimate \(|V(z_1, \ldots, z_N)|\) from below we replace all \(l_k\) in \(\lambda\) by \(h_k\) for \(k = n, n + 1, \ldots, n + s - 1,\) since the distance between any two points \(z_i, z_j\) that belong to the same basic interval of the length \(l_k\), but do not belong to an interval of the length \(l_{k+1}\), is not smaller than \(h_k\).

Therefore,

\[ |V(z_1, \ldots, z_N)| > (l_{n+s} h_{n+s-1}^2 \cdots h_n^{2s})^{2^s} = \lambda^{2^s} \cdot a \]
with \( \log a = 2^s (2 \log \frac{\ln n + s + 1}{\ln n + s - 1} + \cdots + 2^s \log \frac{\ln n}{\ln n - 1}) \). By condition, \( \frac{h_k}{l_k} = 1 - 2^{l_{k+1}} > \frac{1}{2} \). From this \( \log \frac{h_k}{l_k} > -4 \frac{l_{k+1}}{l_k} \geq -4 l_0 \) and \( \log a > -2^s \sum_{k=n}^{n+s-1} 2^{n+s-k+2} l_0^s > -2^{2s+3} l_0^s \), which completes the proof.

**Corollary.**

\[
\frac{1}{2} \sigma_{n,s} \log \frac{1}{n} < I(\mu_N) < \frac{1}{2} \sigma_{n,s} \log \frac{1}{n} + 4 l_0^{c_0}.
\] (1)

**Theorem 3.1.** If the set \( K^{(\alpha_n)} \) has the extension property then

\[
I_n(\mu_{2^{s+1}})/I_n(\mu_{2^s}) \rightarrow 1, \quad \text{as} \quad s \rightarrow \infty, \ n \rightarrow \infty.
\]

Here \( I_n(\mu_{2^{s+1}}) \) stands for the discrete logarithmic energy defined by all endpoints of the type \( s + n \) on any basic interval of the length \( l_n \).

**Proof.** Write \( \gamma_n = 8 l_0^{c_0} \log^{-1} l_0^{-1} \). From (1) we have

\[
\frac{I_n(\mu_{2^{s+1}})}{I_n(\mu_{2^s})} < \frac{\sigma_{n,s} + \gamma_n}{\sigma_{n,s-1}} < \frac{\sigma_{n,s}}{\sigma_{n,s-1}} + \gamma_n,
\]

since \( \sigma_{n,s-1} > 1 \). Now the result follows on the condition (iii) and decrease of the sequence \( \gamma_n \).

One can conjecture that the existence of a linear continuous extension operator for the space \( E(K) \) (at least for Cantor-type sets) is characterized by a regularity of growth of the minimal discrete logarithmic energies corresponding to compact sets that locally form the set \( K \). The points \( (z_k)_{k=1}^{N} \) considered in Lemma give rather rough approximation of the minimal energy for the set \( K^{(\alpha_n)} \cap [0, l_n] \). The exact position of the Fekete points is not known even for rarefied Cantor-type sets.

**References**