(1) Compute the tangents to the hyperbola $X^2 - Y^2 = 1$ and to the parabola $Y = X^2$ (over the real numbers) at their points at infinity. Use the insight gained to give a definition of the concept of an asymptote for algebraic curves defined over arbitrary (e.g. finite) fields.

The hyperbola has two points $[1 : 1 : 0]$ and $[1 : -1 : 0]$ at infinity for any field of characteristic $\neq 2$. The tangents at infinity are $Y - X = 0$ and $Y + X = 0$, which happen to be the asymptotes of the hyperbola over the reals.

The parabola has one point at infinity, namely $[0 : 1 : 0]$. The tangent there is the line at infinity $Z = 0$.

We could define an asymptote as a tangent at infinity that is different from the line at infinity.

(2) Let $K$ be a field of characteristic $\neq 2$, and $f \in K[X]$ a polynomial of degree $\geq 4$ without multiple roots. Show that the projective closure of the hyperelliptic curve $y^2 = f(x)$ has exactly one singular point.

Assume that $r = \deg g \geq 3$. Then the projective closure of $C$ has the equation $F(X, Y, Z) = Y^2 Z^{r-2} - G(X, Z) = 0$. For determining its points at infinity, put $Z = 0$; then $0 = F(X, Y, 0) = -G(X, 0) = -a_r X^r$, hence $X = 0$. Thus $P = [0 : 1 : 0]$ is the only point at infinity on $C^\#$.

We claim that $P$ is singular if $r \geq 4$. We get

$$F_X(P) = -\frac{dG}{dX}(0, 0) = 0,$$

$$F_Y(P) = 2YZ^{r-2}|_P = 0.$$

Now $F_Z = (r - 2)Y^2 Z^{r-3} - \frac{dG}{dZ}(0, 0)$; the last term is 0, hence $F_Z(P) = 0$ if and only if $r > 3$.

It is easily checked that there is no singular point if $r \leq 2$.

(3) Let $f, g, h \in K[x, y]$ be polynomials, and put $f = gh$. Show that any point of intersection of the curves $g(x, y) = 0$ and $h(x, y) = 0$ is a singular point of the curve $f(x, y) = 0$.

Working projectively, we have to show that the partial derivatives of $F$ vanish for points of intersection. But $F_X = G_X H + GH_X$, and plugging in a point $P = [x : y : z]$ satisfying $G(x, y, z) = H(x, y, z) = 0$ we see that $F_X(P) = 0$. The other derivatives also vanish there.
(4) Show that the Klein quartic
\[X^3Y + Y^3Z + Z^3X = 0\]
defined over a field \(K\) is smooth if and only if \(K\) has characteristic \( \neq 7 \).

We first compute derivatives:
\[F_X = 3X^2Y + Z^3,\]
\[F_Y = 3Y^2Z + X^3,\]
\[F_Z = 3Z^2X + Y^3.\]

If \(K\) has characteristic 7, then \([1 : 2 : 4]\) is easily checked to be a singular point.

Now assume that \(K\) has characteristic \( \neq 7 \); then for points at which all three derivatives vanish we have
\[XF_X = 3X^3Y + XZ^3 = -9Y^3Z + XZ^3 = 27XZ^3 + XZ^3 = 28Z^3 = 0,\]
and for fields of characteristic \( \neq 2, 7 \) singular points satisfy \(Z = 0\). By symmetry, we also must have \(X = Y = 0\), hence there is no singular point in these cases.

Assume that \(K\) has characteristic 2; from \(XF_X = X^3Y + XZ^3 = Y^3Z\) we get \(Y = 0\) or \(Z = 0\); this immediately leads to \(X = Y = Z = 0\), hence the Klein quartic is smooth over fields of characteristic 2.

(5) Determine the number of points at infinity of the projective closure of the unit circle \(x^2 + y^2 = 1\) over the finite fields \(F_3, F_5, F_9\).

The points at infinity \([x : y : 0]\) satisfy \(x^2 + y^2 = 0\). There is no solution over \(F_3\); over \(F_5\) there are the points \([1 : 2 : 0]\) and \([1 : 3 : 0]\); finally, writing \(F_9 = F_3(i)\) we find that the points at infinity are \([1 : i : 0]\) and \(1 : -i : 0]\).

Observe that points at infinity are on the line at infinity, and lines intersect conics in at most two points.

(6) Consider the parabola \(C : y = x^2\) over some ring \(R\). Show that the geometric group law defined for conics (with neutral element \(N = (0,0)\)) specializes to
\[(x_1, y_1) + (x_2, y_2) = (x_3, y_3) \quad \text{for} \quad x_3 = x_1 + x_2.\]
Deduce that \(C(R) \simeq (R, +)\), the additive group of \(R\).

Assume first that \(x_1 \neq x_2\). Then the slope of the line through the two points is given by \(m = \frac{y_2 - y_1}{x_2 - x_1} = x_2 + x_1\), and the parallel through \(N = (0,0)\) is \(y = (x_2 + x_1)x\). Now \(x_3\) is the nonzero root of the quadratic \(x^2 = (x_2 + x_1)x\), i.e., \(x = x_2 + x_1\). The cases where \(x_1 = x_2\) are equally simple.

Checking that \(\phi : (x, y) \mapsto x\) induces an isomorphism \(C(R) \longrightarrow (R, +)\) is trivial.

(7) Consider the hyperbola \(C : xy = 1\) over some ring \(R\). Show that the geometric group law defined for conics (with neutral element \(N = (1,1)\)) specializes to
\[(x_1, y_1) + (x_2, y_2) = (x_3, y_3) \quad \text{for} \quad x_3 = x_1x_2.\]
Deduce that \(C(R) \simeq R^\times\), the unit group of \(R\).
Again let me assume that \( x_1 \neq x_2 \); then the slope is \( m = \frac{y_2 - y_1}{x_2 - x_1} = -\frac{1}{x_1 x_2} \in \mathbb{R}^\times \), the parallel through \( N \) is \( y = m(x - 1) + 1 \), hence \( 1 = xy = x[m(x - 1) + 1] \) or \( (x - 1)(mx + 1) = 0 \); the second point of intersection of \( x = -\frac{1}{m} = x_1 x_2 \).

Clearly the map \( C(\mathbb{R}) \to \mathbb{R}^\times : (x, y) \mapsto x \) is an isomorphism.

Note: Never use the quadratic formula to compute the root of a quadratic equation if you already know one solution!