LINEAR ALGEBRA I

THE STORY SO FAR

1. VECTOR SPACES

Informally, a real vector space consists of a set of elements that you can add and multiply by real constants called scalars – the exact set of axioms can be found in any book on linear algebra.

If $V$ is a vector space and if $U$ is a subset of $V$ then we say that $U$ is a subspace of $V$ if $U$ is also a vector space. Example: the vector space of polynomials of degree $\leq 2$ is a subspace of the vector space of polynomials of degree $\leq 3$.

A linear combination of vectors $v_1, \ldots, v_n$ is an expression $a_1v_1 + \ldots + a_nv_n$ with real constants $a_1, \ldots, a_n$.

The span of vectors $v_1, \ldots, v_n$ from some vector space $V$ is the set of all linear combinations $a_1v_1 + \ldots + a_nv_n$. Theorem: span($v_1, \ldots, v_n$) is a subspace of $V$.

Vectors $v_1, \ldots, v_n$ are called linearly dependent if there is a nontrivial solution of $a_1v_1 + \ldots + a_nv_n = 0$. If $a_1 = \ldots = a_n = 0$ is the only solution, they are called linearly independent.

2. EXAMPLES

(1) Examples of real vector spaces: the set of all vectors $(x, y)$ with $x, y \in \mathbb{R}$ (similarly for higher “dimension”, the set of all polynomials, the set of all continuous functions $\mathbb{R} \rightarrow \mathbb{R}$, the set of solutions of the differential equation $y' = y$ on $\mathbb{R}$, and a whole lot more.

(2) Write $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ as a linear combination of $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \end{pmatrix}$.

From $\begin{pmatrix} 1 \\ -2 \end{pmatrix} = a\begin{pmatrix} -1 \\ 3 \end{pmatrix} + b\begin{pmatrix} 1 \end{pmatrix}$ we get the linear equations $3 = -a + b$ and $2 = 2a + b$. Solving this system shows $a = -\frac{1}{4}$ and $b = \frac{3}{2}$.

(3) Are the polynomials $x^2 - 1$, $x + 1$, $x^2 + x$ linearly independent in the vector space of polynomials of degree $\leq 2$?

Write $a(x^2 - 1) + b(x + 1) + c(x^2 + x) = 0$ for scalars $a, b, c \in \mathbb{R}$. This implies $(a + c)x^2 + (b + c)x + (b - a) - 1 = 0$. Now a polynomial is the 0-element if and only if all of its coefficients are 0, hence we see that $a + c = 0$, $b + c = 0$ and $b - a = 0$. Eliminating $c$ we get $a - b = 0$ (twice), hence $a = b = 1$ and $c = -1$ is a nonzero solution, i.e., $(x^2 - 1) + (x + 1) = (x^2 + x)$ is a nontrivial relation between these polynomials. In particular, span$(x^2 - 1, x + 1, x^2 + x) = \text{span}(x^2 - 1, x + 1) = \text{span}(x + 1, x^2 + x) = \text{span}(x^2 - 1, x^2 + x)$.

(4) Assume that the vectors $v_1, v_2, v_3$ are linearly independent in $U$, and that $U$ is a subspace of $V$. Are $v_1, v_2, v_3$ linearly independent in $V$?

If $v_1, v_2, v_3$ were linearly dependent in $V$, then there would be a nontrivial relation $a_1v_1 + a_2v_2 + a_3v_3 = 0$ in $V$. But this is also a relation in $U$, so $v_1, v_2, v_3$ would also be linearly dependent in $U$, which they are not.

Thus the answer to the question is “yes”.

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