(1) True or False? (No explanation required)

<table>
<thead>
<tr>
<th>statement</th>
<th>true</th>
<th>false</th>
</tr>
</thead>
<tbody>
<tr>
<td>Every nonzero matrix $A$ has an inverse $A^{-1}$</td>
<td>×</td>
<td></td>
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<tr>
<td>The inverse of a product $AB$ of square matrices $A, B$ is equal to $A^{-1}B^{-1}$</td>
<td></td>
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<tr>
<td>Homogeneous linear systems of equations always have a solution</td>
<td></td>
<td>×</td>
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<tr>
<td>The rank of an $m \times n$-matrix is always $\leq n$</td>
<td></td>
<td>×</td>
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<tr>
<td>The set of polynomials of degree $= 2$ is a vector space</td>
<td></td>
<td>×</td>
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<tr>
<td>The product of an $m \times n$- and a $n \times k$-matrix is an $m \times k$-matrix</td>
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<tr>
<td>If ${v_1, v_2, v_3}$ is a basis of a vector space, then ${v_1, v_2, v_3}$ are linearly independent</td>
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<tr>
<td>If ${v_1, v_2, v_3}$ are linearly independent vectors in some vector space $V$, then they form a basis of $V$</td>
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<tr>
<td>The row rank of a matrix is equal to its column rank</td>
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<td>×</td>
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<tr>
<td>For all $2 \times 2$-matrices $A$ and $B$, we have $AB = BA$</td>
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</table>

Explanations: matrices like \((\begin{smallmatrix}1 & 0 \\ 0 & 0\end{smallmatrix})\) or \((\begin{smallmatrix}1 & 1 \\ 1 & 1\end{smallmatrix})\) are nonzero but do not have an inverse. Matrices have an inverse if and only if they are nonsingular square matrices.

If $A$ and $B$ are nonsingular, then so is $AB$, and its inverse clearly is $B^{-1}A^{-1}$ since $B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I$. In general, $B^{-1}A^{-1} \neq A^{-1}B^{-1}$ since matrix multiplication is not commutative.

Homogeneous systems $Ax = 0$ always have the solution $x = 0$.

Since the matrix has at most $n$ columns, the column rank is at most $n$.

The set of polynomials of degree $= 2$ is not a vector space since it does not contain 0.

A basis of $V$ is a set of linearly independent vectors that span $V$. Thus basis vectors are always linearly independent, but not every set of linearly independent vectors form a basis: for example, \((\begin{smallmatrix}1 \\ 0\end{smallmatrix})\) is linearly independent, but does not form a basis of $V = \mathbb{R}^2$. 


(2) Compute the solution space of the homogeneous system $Ax = 0$ for

$$A = \begin{pmatrix} 2 & -1 & -2 \\ -4 & 2 & -4 \\ -8 & 4 & 8 \end{pmatrix}.$$  

What is the rank of $A$?

If I ask you to compute the solution space, then your job is to compute the solution space. It is not sufficient to just compute the rank of $A$.

For solving the system of equations, we perform row operations:

$$\begin{pmatrix} 2 & -1 & -2 & 0 \\ -4 & 2 & -4 & 0 \\ -8 & 4 & 8 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & -2 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus the solutions are $x_3 = 0$, $x_2 = r$, $x_1 = r/2$, hence the solution space is the span of $\{(1/2, 0)\}$. Moreover, the (column) rank is obviously equal to 2; alternatively, the rank is 3 minus the dimension 1 of the solution space.

(3) For which values of $a$ does the inverse $A^{-1}$ of

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & a \end{pmatrix}$$

exist? Compute $A^{-1}$ in these cases.

$$\begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 2 & a & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 2 & a & 0 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & a & 2 & 1 & 1 \end{pmatrix}$$

where in the last step we have assumed that $a \neq 0$. In fact, if $a = 0$ then the matrix is singular and does not have an inverse; if $a \neq 0$, $A^{-1}$ exists and is given by

$$A^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 0 \\ -2/a & 1/a & 1/a \end{pmatrix}.$$
(4) Are the “vectors” \((\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}), (\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}), (\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix})\) in the real vector space \(M_{22}\) of \(2 \times 2\)-matrices linearly independent? We have to solve the system of equations
\[ a(\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}) + b(\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}) + c(\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}) = 0. \]
This gives us the linear system of equations represented by
\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1 \\
1 & 0 & 0 \\
1 & 2 & 2
\end{pmatrix}
\]
which is easily solved. We get \(a = b = c = 0\) as the unique solution, therefore these matrices are linearly independent.

(5) Let \(P\) be an \(n \times n\)-matrix with \(P^2 = P\), let \(I\) denote the identity matrix of dimension \(n\), and let \(w \in \mathbb{R}^n\) be an arbitrary vector. Show that every vector \(v = (P - I)w\) is a solution of the homogeneous system \(Pv = 0\).

All you needed to do was check that \(Pv = 0\). But this is easy: \(Pv = P(P - I)w = (P^2 - P)w = (P - P)w = 0\).

(6) a) Find a basis for the vector space of all polynomials \(p\) of degree \(\leq 3\) with \(p(0) = p'(1) = 0\). Let \(p(x) = ax^3 + bx^2 + cx + d\); then \(0 = p(0) = d\) and \(0 = p'(1) = 3a + 2b + c\). Thus the polynomials in \(V\) have the form \(p(x) = ax^3 + bx^2 - (3a + 2b)x\), and a basis is given by \(\{x^3 - 3x, x^2 - 2x\}\) (these polynomials span \(V\), and they are clearly independent since they have distinct degrees).

b) Write \(p(x) = x^3 + 2x^2 - 7x\) as a linear combination of your basis. Obviously \(p(x) = 1(x^3 - 3x) + 2(x^2 - 2x)\).