ALGEBRAIC GEOMETRY

HOMEWORK 1

Due Tu 15.02.04

(1) Find all points on the following curves with coordinates in the finite fields $F_4$ and $F_5$:

- the line $x - 2y + 1 = 0$;
- the 'circle' $x^2 + y^2 + 1 = 0$.

- the line $x - 2y + 1 = 0$;
  In $F_4$ we have $2 = 0$, hence the equation of the line is $x + 1 = 0$; there are exactly 4 such points, namely $(1 : a)$ for $a \in F_4$.
  In $F_5$, every value of $y$ will give exactly one value of $x$, namely $x = 2y - 1$. Thus the line consists of the 5 points $(-1, 0)$, $(1, 1)$, $(3, 2)$, $(0, 3)$ and $(2, 4)$.

- the 'circle' $x^2 + y^2 + 1 = 0$.
  Let $F_4 = \{0, 1, a, b\}$. Plugging in all possible values for $x$ and solving for $y$ will provide you with a correct answer. Here’s a more tricky solution: since $2 = 0$, we have $x^2 + y^2 + 1 = (x + y + 1)^2$, hence the 'circle' is nothing but the double line $x + y + 1 = 0$, and its points are $(0, 1)$, $(1, 0)$, $(a, b)$ and $(b, a)$.
  Now consider $F_5$; here brute force shows that the points on this curve are $(0, \pm 2)$, $(\pm 1, 0)$, and $(\pm 2, 0)$.

(2) Determine the rational points on the hyperbola $X^2 - 3Y^2 = 1$ with as many methods as possible.

1. Geometric Method: start with $P = (-1, 0)$, consider lines through $P$ with rational slope $t$, and compute the second point of intersection by factoring out $(x + 1)$. Then $x = \frac{1 + 3t^2}{1 - 3t^2}$ and $y = \frac{-2t}{1 - 3t^2}$.

2. Algebraic Method: Clear denominators; then you get $a^2 - 3b^2 = c^2$ for coprime integers $a, b, c$. Factor the equation as $(a - c)(a + c) = 3b^2$. Either $a$ or $b$ is even, thus there are two cases:
   a) $a$ is even; then $b$ and $c$ are odd. Since $\gcd(a - c, a + c) \mid \gcd(2a, 2c) = 2$, the gcd must be 1 or 2; since $a$ is even and $c$ is odd, the gcd is 1. Unique factorization gives $a - c = r^2$, $a + c = 3s^2$ or $a - c = 3r^2$, $a + c = s^2$, but replacing $c$ by $-c$ allows us to assume that the first choice holds. Then $2a = r^2 + 3s^2$, $2c = 3s^2 - r^2$, and $b = rs$, and the last equation shows that $r$ and $s$ are both odd.

b) $a$ is odd; then $b$ must be even, as a little computation mod4 shows. Here we find $\gcd(a - c, a + c) = 2$, and as above this shows $a - c = 2r^2$, $a + c = 6s^2$, hence $a = r^2 + 3s^2$, $b = 2rs$, and $c = 3s^2 - r^2$. 

3. Galois Theory. If \( a^2 = c^2 + 3b^2 \), then \( \alpha = \frac{c+bf\sqrt{-3}}{a} \) has norm 1, so by Hilbert 90 we get \( \alpha = \frac{r+sf\sqrt{-3}}{r-3s} \). Comparing real and imaginary parts then yields the formulas \( x = \frac{r^2+3s^2}{r^2-3s^2}, y = \frac{2rs}{r^2-3s^2} \).

(3) Show that the curve \( X^{1/3} + Y^{1/3} = Z^{1/3} \) is a plane algebraic curve.

Raising the equation to the third power gives
\[
Z = X + 3X^{2/3}Y^{1/3} + 3X^{1/3}Y^{2/3} + Y
= X + Y + 3X^{1/3}Y^{1/3}(X^{1/3} + Y^{1/3})
= X + Y + 3X^{1/3}Y^{1/3}Z^{1/3},
\]
hence \( (Z - X - Y)^3 = 27XYZ \).

What this shows is that any point satisfying the original equation lies on the plane algebraic curve \( (Z - X - Y)^3 = 27XYZ \). The converse, however, is not so clear, because the cube roots pose a problem, in particular over \( \mathbb{C} \) or over finite fields. The moral of the story is: don’t use roots in algebraic geometry unless you really really have to.

(4) Use the sweeping line technique to parametrize the conic \( x^2 - y^2 + 2x + 1 = 0 \) using
(a) \( P = (0, 1) \)
(b) \( Q = (-1, 0) \)
as your starting point. Explain your observations (if you can’t, use sing surf to sketch the curve).

Actually, parametrization seems to work with \( P \) because you get \( x = \frac{-2}{t^2+1} \) and \( y = \frac{1-t^2}{t^2+1} \). Using \( Q \), on the other hand, will end in disaster. The reason is that the conic in this problem is degenerate: it is a pair of lines intersecting in \( Q \). Thus in b) you only find the point you start with, and in a) you only find the points on one of the two lines.