Consider a Cartesian coordinate system $S$ with unit vectors $\hat{i}, \hat{j},$ and $\hat{k}$ directed along the three axes $x, y,$ and $z,$ respectively. Any vector $\vec{A}$ in this system can be expressed as

$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$  \hspace{1cm} \text{(1)}$$

where its Cartesian components are given by

$$A_x = \vec{A} \cdot \hat{i}, \quad A_y = \vec{A} \cdot \hat{j}, \quad A_z = \vec{A} \cdot \hat{k}$$

Suppose that we rotate this system about the $z$ axis through an angle $\vartheta$. The new set of axes in the rotated frame $S'$ are $x', y'$, and $z'$; where the corresponding set of new unit vectors denoted by $\hat{i}', \hat{j}',$ and $\hat{k}'$.

To set up the relations between the primed unit vectors $\{\hat{i}', \hat{j}', \hat{k}'\}$ and $\{\hat{i}, \hat{j}, \hat{k}\}$ we refer to eq.(1), where we replace the "any vector $\vec{A}$" therein successively by $\hat{i}', \hat{j}',$ and $\hat{k}'$. Letting $\vec{A}$ stand for $\hat{i}'$, for instance, we write

$$\hat{i}' = (\hat{i}' \cdot \hat{i}) \hat{i} + (\hat{i}' \cdot \hat{j}) \hat{j} + (\hat{i}' \cdot \hat{k}) \hat{k}$$

Similarly, we obtain

$$\hat{j}' = (\hat{j}' \cdot \hat{i}) \hat{i} + (\hat{j}' \cdot \hat{j}) \hat{j} + (\hat{j}' \cdot \hat{k}) \hat{k}$$

and

$$\hat{k}' = (\hat{k}' \cdot \hat{i}) \hat{i} + (\hat{k}' \cdot \hat{j}) \hat{j} + (\hat{k}' \cdot \hat{k}) \hat{k}$$

Noting that

$$\hat{i}' \cdot \hat{i} = \hat{j}' \cdot \hat{j} = \cos \vartheta, \quad \hat{i}' \cdot \hat{j} = \cos(\pi/2 - \vartheta), \quad \hat{j}' \cdot \hat{i} = \cos(\pi/2 + \vartheta)$$

we write

$$\hat{i}' = \cos \vartheta \hat{i} + \sin \vartheta \hat{j}$$

$$\hat{j}' = -\sin \vartheta \hat{i} + \cos \vartheta \hat{j}$$

$$\hat{k}' = \hat{k}$$
Note that vector $\vec{A}$, eq.(1), having components $A_x, A_y, A_z$ in the original system $S$ should look somewhat different with modified components $A_x', A_y', A_z'$ as viewed in the new rotated frame $S'$, i.e., one writes

$$\vec{A}' = A_x' \hat{i}' + A_y' \hat{j}' + A_z' \hat{k}'$$

Actually, this vector does not change neither in magnitude nor in direction. Whether viewed in $S$ or $S'$, it remains the same. What makes up the difference is the orientation of the coordinate axes. Thus, having $\vec{A}' = \vec{A}$, we write

$$A_x' \hat{i}' + A_y' \hat{j}' + A_z' \hat{k}' = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

Forming a scalar product of each of the six terms in the above equation by $\hat{i}'$ one obtains

$$A_x' = A_x (\hat{i}' \cdot \hat{i}) + A_y (\hat{i}' \cdot \hat{j})$$

$$= A_x \cos \vartheta + A_y \sin \vartheta$$

Similarly one gets

$$A_y' = A_x (\hat{j}' \cdot \hat{i}) + A_y (\hat{j}' \cdot \hat{j})$$

$$= A_x (- \sin \vartheta) + A_y \cos \vartheta$$

and

$$A_z' = A_z$$

Using the above relations one can easily show that

$$A_x'^2 + A_y'^2 + A_z'^2 = A_x^2 + A_y^2 + A_z^2$$

Thus, the conclusion reached here is that the magnitude of $\vec{A}$ remains invariant under a rotation of coordinate axes. In a similar manner one should also expect the dot product $\vec{A} \cdot \vec{B}$ to remain the same if the coordinate axes are rotated; simply because the dot product of any two vectors is a scalar.