Math 240-01, Spring 2000

Solutions to the Final Exam

Problem 1. A 1-kg weight is attached to the lower end of a spring suspended from the ceiling. In its equilibrium position the weight stretches the spring 10 cm. Starting at time \( t_0 = 0 \) the weight is pulled down by a force of 30 N for \( \pi \) sec and then released. Find the displacement of the weight as function of time and the amplitude of the resulting free oscillation. (Assume \( g = 10 \).)

Solution: Introduce the constants: \( m = 1 \) kg is the mass of the weight, \( y_0 = 0.1 \) m, the original displacement in the equilibrium position, \( k \), the spring coefficient, \( F_0 = 30 \) N, the value of the force applied to the weight, and \( a = \pi \) sec, the duration of the force. From Hook’s law we have \( mg = ky_0 \), i.e., \( k = mg/y_0 \). The force applied as function of time can be given by \( F(t) = F_0(1 - \alpha(a - t)) \), where \( \alpha \) is the step function. Newton’s second law gives the equation \( my'' = -k(y + y_0) + mg + F(t) \), \( y(0) = 0 \), \( y'(0) = 0 \), or \( y'' + (k/m)y = (F_0/m)(1 - \alpha(t - a)) \). (Here \( y \) is the displacement; the axis is directed straight down and the origin is chosen at the equilibrium position.) Let \( \hat{y} = L\{y\} \). Apply the Laplace transform:

\[
    s^2 \hat{y} + \frac{k}{m} \hat{y} = \frac{F_0}{ms} (1 - e^{-as}), \quad \text{or} \quad \hat{y} = \frac{F_0/m}{s(s^2 + (k/m))} (1 - e^{-as}).
\]

Hence (using properties of \( L^{-1} \)), \( y = G(t) - G(t - a)\alpha(t - a) \), where \( G(t) \) is the inverse transform of \( g(s) = F_0/[ms(s^2 + (k/m))] \). To find \( G(t) \), use the tables:

\[
    g(s) = \frac{F_0}{k} \left[ \frac{1}{s} - \frac{s}{s^2 + (k/m)} \right], \quad \text{whence} \quad G(s) = \frac{F_0}{k} \left( 1 - \cos \sqrt{\frac{k}{m}} t \right).
\]

Finally, substituting the constants, we get

\[
    y(t) = 0.3(1 - \alpha(t - \pi)) - 0.3(\cos 10t - \cos 10(t - \pi)\alpha(t - \pi))
\]

When the force is removed (i.e., \( t > \pi \)), one has \( \alpha(t - \pi) = 1 \) and, since \( \cos 10(t - \pi) = \cos 10t \), we get \( y(t) = 0 \) for \( t > \pi \), i.e., the weight comes back to the equilibrium position and there is no oscillation! (Thus, the amplitude is \( \sqrt{1} \).) Surprising, isn’t it?

Problem 2. Represent the general solution to \( (x^2 - 2)y'' + 6xy' + 6y = 0 \) by a power series about the origin. Indicate the interval of convergence of the series.

Solution: Let \( y = \sum_{n=0}^{\infty} c_n x^n \) be the series in question. Plug in to get

\[
    \sum_{n=0}^{\infty} n(n - 1)c_n x^n - 3 \sum_{n=0}^{\infty} n(n - 1)c_n x^{n-2} + 6 \sum_{n=0}^{\infty} n c_n x^n + 6 \sum_{n=0}^{\infty} c_n x^n = 0,
\]

or, after shifting the index in the second sum, \( \sum_{n=0}^{\infty} [n(n - 1)c_n - 3(n + 2)(n + 1)c_{n+2} + 6nc_n + 6c_n]x^n = 0 \). Equating the coefficients to 0, we get the recurrence relation \( (n^2 + 5n + 6)c_n = 3(n + 2)(n + 1)c_{n+2} \), or \( c_{n+2} = (n + 3)c_n/3(n + 1) \). Shift the index back: \( c_n = (n + 1)c_{n-2}/3(n - 1), n \geq 2 \). This obviously yields

\[
    c_{2k} = \frac{(2k + 1)(2k - 1)\ldots(3)}{3^k(2k - 1)(2k - 3)\ldots(1)} c_0 = \frac{2k + 1}{3^k} c_0 \quad \text{and} \quad c_{2k+1} = \frac{(2k + 2)(2k - 2)\ldots(4)}{3^k(2k - 2)(2k - 4)\ldots(2)} c_1 = \frac{k + 1}{3^k} c_1.
\]

Thus, the solution is

\[
    y = c_0 \sum_{k=0}^{\infty} \frac{2k + 1}{3^k} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{k + 1}{3^k} x^{2k+1}.
\]

The equation has two singular points, \( x = \pm \sqrt{3} \). Hence, the interval of convergence of the series is \(( -\sqrt{3}, \sqrt{3} )\).
Problem 3. Find the inverse Laplace transforms of

(a) \( f(s) = \frac{2s^2 - 2}{s^2(s^2 - 2s + 2)} \), \quad (b) \( f(s) = \frac{1}{s^2 + 6s + 13} \).

Solution: (a) Since \( s^2 - 2s + 2 = (s - 1)^2 + 1 \) has no real roots, the elementary fraction decomposition has the form

\[
\frac{2s^2 - 2}{s^2(s^2 - 2s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs + D}{(s^2 - 2s + 2)}.
\]

Get rid of the denominators, \( 2s^2 - 2 = As(s^2 - 2s + 2) + B(s^2 - 2s + 2) + (Cs + D)s^2 \), and equate the coefficients: \( A + C = 0 \), \(-2A + B + D = 2 \), \( 2A - 2B = 0 \), \( 2B = -2 \). Hence, \( A = B = -1 \), \( C = D = 1 \), and

\[
L^{-1}\left\{ \frac{2s^2 - 2}{s^2(s^2 - 2s + 2)} \right\} = -L^{-1}\left\{ \frac{1}{s} \right\} - L^{-1}\left\{ \frac{1}{s^2} \right\} + L^{-1}\left\{ \frac{(s - 1) + 2}{(s - 1)^2 + 1} \right\} = \frac{-1 - t + e^t \cos t + 2e^t \sin t}{2}.
\]

(b) This one is straightforward:

\[
L^{-1}\left\{ \frac{1}{s^2 + 6s + 13} \right\} = L^{-1}\left\{ \frac{1}{(s + 3)^2 + 4} \right\} = \frac{1}{2}e^{-3t} \sin 2t.
\]

Problem 4. Find the first five terms of the power series expansion about \( x_0 = 1 \) of the solution to

\[ y'' = (y')^3 + x^2, \quad y(1) = 1, \quad y'(1) = 1. \]

Solution: We need the values of \( y \) and its first four derivatives at 1. From the initial conditions we have \( y(1) = 1 \) and \( y'(1) = 1 \), and, differentiating the equation and plugging in, we obtain:

\[
\begin{align*}
y'' &= (y')^3 + x^2, \\
y''' &= 3(y')^2 y'' + 2x, \\
y^{IV} &= 6y'(y'')^2 + 3(y')^2 y''' + 2,
\end{align*}
\]

Thus, \( y = \sum_{n=0}^{\infty} y^{(n)}(1)(x - 1)^n/n! = 1 + (x - 1) + (x - 1)^2 + \frac{4}{3}(x - 1)^3 + \frac{25}{12}(x - 1)^4 + \ldots \).

Problem 5. Show that

\[
L^{-1}\left\{ \frac{1}{\sqrt{s(s - 1)}} \right\} = \frac{2e^t}{\sqrt{\pi}} \int_0^\sqrt{t} e^{-x^2} dx.
\]

Solution: We have \( L^{-1}\{1/\sqrt{s}\} = 1/\sqrt{\pi t} \) and \( L^{-1}\{1/(s - 1)\} = e^t \). Use the convolution theorem:

\[
L^{-1}\left\{ \frac{1}{\sqrt{s(s - 1)}} \right\} = \frac{1}{\sqrt{\pi}} \int_0^t e^{-u} \frac{du}{\sqrt{u}} = \frac{2e^t}{\sqrt{\pi}} \int_0^{\sqrt{t}} e^{-x^2} dx \quad \text{(substituting } u = x^2). \]