Solutions to Final Exam

Problem 1. If possible, diagonalize the matrix and find an orthogonal basis in which it has diagonal form:

\[ A = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 5 & 4 \\ -4 & 4 & 3 \end{bmatrix}. \]

Solution: The characteristic polynomial is \( f_A(x) = x^3 - 9x^2 - 9x + 81. \) By trial and error one finds that \( x = 3 \) is a root, and the division gives \( f_A(x) = (x - 3)(x^2 - 6x - 27). \) Hence, the other two roots are \(-3\) and \(9. \) All roots are real and simple, and the matrix is diagonalizable. The eigenvectors are found from the corresponding systems, and final answer is

\[ B = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{bmatrix} \quad \text{in the basis} \quad \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}. \]

The requirement to find an orthogonal basis was intentionally misleading. When all eigenvalues are distinct, the basis is rigid (one can only multiply vectors by numbers); on the other hand, since the original matrix is symmetric, the basis found is automatically orthogonal!

Problem 2. Find \( A^{-1} \) for

\[ A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 4 & 0 & 1 & -2 \end{bmatrix}. \]

Solution: I didn’t mean you to use the discriminant formula for \( A^{-1}, \) though you can always use any way you prefer. Anyway, here is the answer:

\[ A^{-1} = \begin{bmatrix} -4 & -3 & -2 & -1 \\ -7 & -6 & -4 & -2 \\ -10 & -8 & -6 & -3 \\ -13 & -10 & -7 & -4 \end{bmatrix}. \]

Problem 3. Evaluate

\[
\begin{bmatrix}
1 & 2 & 2 & 0 & 3 \\
0 & 1 & 2 & 0 & -1 \\
2 & 3 & 5 & 2 & 1 \\
1 & 1 & 2 & 1 & 1 \\
1 & 3 & 3 & 0 & 4
\end{bmatrix}.
\]

Solution: The calculation is straightforward (simplifying the matrix via elementary transformations and using row/column expansion), and the correct answer is \(-2\).

Problem 4. Let \( A \) be an \((n \times n)\)-matrix. Show that, if \( \det(A - \lambda I_n) = 0 \) for some \( \lambda \in \mathbb{R}, \) then \( \det(A^2 - \lambda^2 I_n) = 0. \)

Solution: First approach: Note that \( A^2 - \lambda^2 I_n = (A - \lambda I_n)(A + \lambda I_n) \) (as \( A \) commutes with \( I_n \) and \( I_n^2 = I_n \)). Then \( \det(A^2 - \lambda^2 I_n) = \det(A - \lambda I_n)\det(A + \lambda I_n) = 0. \)

Second approach: The hypotheses mean that \( \lambda \) is an eigenvalue of \( A, \) i.e., there is a vector \( x \neq 0 \) such that \( Ax = \lambda x. \) Then \( A^2 x = A(\lambda x) = \lambda^2 x, \) i.e., \( \lambda^2 \) is an eigenvalue of \( A^2. \)

Problem 5. Let \( P_3 \) be the space of polynomials of degree up to 3 with the inner product

\[ (p, q) = p(1) \cdot q(1) + p'(1) \cdot q'(1) + p''(1) \cdot q''(1) + p'''(1) \cdot q'''(1) \]

and \( W = \text{Span}\{1, t^2\}. \) Find a basis for \( \text{Ker proj}_W. \)

Solution: First, note that \( \text{Ker proj}_W = W^\perp \) (why?). Hence, it is given by the equations \((p, 1) = (p, t^2) = 0. \) The first one gives \( p(1) = 0, \) the second one, \( p(1) + 2p'(1) + 2p''(1) = 0. \) In the basis \( \{1, (t - 1), (t - 1)^2, (t - 1)^3\} \) the solution is immediate: \( p = \alpha(t - 1)^3 + \beta(t - 1)^2 - 2(t - 1). \) Hence, one can take for a basis \( \{(t - 1)^3, (t - 1)^2 - 2(t - 1)\}. \)

Remark: Writing down a formula for \( \text{proj}_W \) and then finding its kernel is a lot of work! A common mistake was ignoring the fact that for the formula one needs an orthogonal basis in \( W. \)