Solutions to Final Exam

Problem 1. If possible, diagonalize the matrix and find an orthogonal basis in which it has diagonal form:

\[ A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}. \]

Solution: The characteristic polynomial is

\[
\det \begin{bmatrix} -1 - \lambda & 2 & 2 \\ 2 & -1 - \lambda & 2 \\ 2 & 2 & -1 - \lambda \end{bmatrix} = \lambda^3 + 3\lambda^2 - 9\lambda - 27 = (\lambda - 3)(\lambda + 3)^2.
\]

(All roots are integers, so they can be found by trial and error, among the divisors of 27.) The corresponding homogeneous systems for eigenvectors and their solutions are:

\[
\lambda = 3: \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} X = 0; \quad u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \\
\lambda = -3: \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} X = 0; \quad u_{2,3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}.
\]

Thus, the matrix can be diagonalized, and it has diagonal form in the basis \( \{u_1, u_2, u_3\} \). Since \( A \) is symmetric, \( u_1 \) is orthogonal to \( u_2, u_3 \), and we only need to orthogonalize \( u_2, u_3 \). (By the way, the fact that \( A \) is symmetric also tells us that it is diagonalizable, i.e., we must find three independent eigenvectors!) Applying Gram-Schmidt to \( \{u_2, u_3\} \), we replace \( u_3 \) with \([-1 2 1]^T\). Finally, the matrix has diagonal form

\[ \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \]

in the orthogonal basis \( \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \right\} \).

Problem 2. Find \( A^{-1} \) for

\[ A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 1 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 \end{bmatrix}. \]

Solution: The problem is straightforward, you can use any way you like. The correct answer is

\[ A^{-1} = \frac{1}{2} \begin{bmatrix} -2 & -2 & -1 & -1 \\ -2 & -4 & -2 & -2 \\ -1 & -2 & -2 & -1 \\ -1 & -2 & -1 & -2 \end{bmatrix}. \]

Problem 3. Let \( A \) be a square matrix with integral entries. Prove that \( A^{-1} \) exists and has integral entries if and only if \( \det A = \pm 1 \).

Solution: Clearly, the determinant of an integral matrix is an integer (as it is obtained from the entries of the matrix by addition and multiplication only). Thus, for the only if part it suffices to notice that \( \det A \det A^{-1} = \det (AA^{-1}) = \det I = 1 \). Since the product of two integers is 1, they must both be \( \pm 1 \). The if part follows from the formula \( A^{-1} = (1/\det A) \text{adj} A \) and the fact that, if \( A \) is integral, so is \( \text{adj} A \) (as its entries are determinants of integral matrices, see above).

Problem 4. Let \( L: P_3 \to P_3 \) be given by \( L[p(t)] = p(t) + p(1)(t-3) - 2p'(1)(t-1) \). Find the Eigenvalues of \( L \), their multiplicities, and the Eigenvectors of \( L \). Is \( L \) diagonalizable?

Solution: Take for a basis the set \( \{(t-1)^3, (t-1)^2, (t-1), 1\} \). Then \( L \) is represented by the matrix

\[ A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \]

and the characteristic polynomial and eigenvalues are found immediately: \( f_A(\lambda) = (\lambda - 1)^2(\lambda + 1)^2 \) has two roots, \( \lambda = 1 \) and \( -1 \), each of multiplicity 2. As a matter of fact, the eigenvectors are also found immediately, as the matrices obtained are already in (almost) row echelon form. A basis for the eigenspace corresponding to \( \lambda = 1 \) is \( \{(t-1)^3, (t-1)^2\} \), and a basis for the eigenspace corresponding to \( \lambda = -1 \) is \( \{(t-1)\} \). Thus, one eigenvector is missing and the operator is not diagonalizable.
Problem 5. Let 

\[ A_1 = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \quad A_4 = \begin{bmatrix} -2 & 1 \\ 1 & 0 \end{bmatrix} \]

be vectors in the space \( M_{2,2} \) of \((2 \times 2)\)-matrices with the inner product \((A, B) = \text{trace}(AB^T)\) and \(W = \text{Span}\{A_1, A_2, A_3, A_4\}\). Find a basis for \(W^\perp\). What are \(\dim W\) and \(\dim W^\perp\)?

Solution: First of all, note that \((A, B) = \text{trace}(AB^T)\) is just the standard inner product \((u, v) = u^Tv\) with respect to a natural basis for \(M_{2,2}\). (A straightforward calculation discussed in class.) Thus, pick a natural basis, say, \(\{E_{11}, E_{12}, E_{21}, E_{22}\}\), represent the given matrices as vectors, and write down the system for \(W^\perp\):

\[
\begin{bmatrix}
2 & 1 & 0 & 4 \\
0 & 1 & 2 & 2 \\
1 & 1 & 1 & 3 \\
-2 & 1 & 1 & 0
\end{bmatrix}
X = 0.
\]

(Note that the given vectors form the columns of the matrix, as they are transposed in the expression for the standard inner product.) The system has 1-dimensional solution space spanned by the vector corresponding to \(\begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix}\). Hence, \(\dim W^\perp = 1\) and \(\dim W = \dim M_{2,2} - \dim W^\perp = 3\). (In particular, this means that the given vectors are linearly dependent. However, there is no need to prove this directly or construct an explicit basis for \(W\).)