ATTENTION! HOMEWORK PROCEDURE/DEADLINE CHANGE

Your dishonesty (missing papers, cheating, etc.) forces me to change the procedure once more. From now on, the homeworks are to be deposited on Wednesday by 9 am sharp at a box located at the entrance to SA building. (In particular, this means that you can do your homework during the tutorial sessions.) The security are instructed to supervise the procedure and to remove the box at 9 am. Sorry for any inconvenience. (Well, you may still cheat, but the donor should be aware of the fact as he/she is going to share the grade.)

Team work vs. cheating! You can do homework together in groups of up to 3 students, provided that this is clearly indicated in your paper and each student represents his/her own understanding of the problem and its solution. Otherwise, identical papers will be rejected.

Unlike quizzes/exams, plenty of time is given for homeworks. Thus, we expect a neatly prepared paper with all your work shown. Always give reasons to your answers (even if it is not requested explicitly in the problem)!

All references are to Thomas & Finney, Calculus and Analytic Geometry, 9th Edition. For your convenience, I am also stating the problems here.

About this page. I'm doing my best to provide you with the solutions to your homework problems. Of course, the ideas presented here are correct :) However, since I'm just typing it in, without seeing much of the output, you should be aware of the fact that I may make some minor arithmetical mistakes once in a while . . .

Assignment 7 (due 14/05 by 9 am): page 496, no. 14; page 496, no. 38; page 496, no. 50.

Problem 1. Use L'Hôpital’s rule to find the limit \( \lim_{x \to 1} \frac{x - 1}{\ln x - \sin(\pi x)} \).

Solution.

\[
\lim_{x \to 1} \frac{x - 1}{\ln x - \sin(\pi x)} = \lim_{x \to 1} \frac{1}{1 - \pi \cos(\pi x)} = \frac{1}{1 + \pi}.
\]

Problem 2. Use L'Hôpital’s rule to find the limit \( \lim_{x \to \infty} \frac{1}{x \ln x \int_1^x \ln t \, dt} \).

Solution.

\[
\lim_{x \to \infty} \frac{1}{x \ln x \int_1^x \ln t \, dt} = \lim_{x \to \infty} \frac{1}{x \ln x} = 1.
\]

Problem 3. Find the limit \( \lim_{x \to 0} (e^x + x)^{1/x} \).

Solution. Let \( \lim_{x \to 0} (e^x + x)^{1/x} = A \). Then

\[
\ln A = \lim_{x \to 0} \frac{e^x + x}{x} = \lim_{x \to 0} \frac{e^x + 1}{1} = 2,
\]

and \( A = e^2 \).

Assignment 6 (due 07/05 by 9 am): page 465, no. 50; page 466, no. 68; page 472, no. 28.

Problem 1. Use logarithmic differentiation to find \( \frac{dy}{dx} \):

\[
y = \sqrt[3]{\frac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}.
\]

Solution. One has \( \ln y = \frac{1}{3} \left( \ln x + \ln(x+1) + \ln(x-2) - \ln(x^2+1) - \ln(2x+3) \right) \), and the formula \( y' = y(\ln y)' \) (and, of course, the chain rule) gives us

\[
y' = \frac{1}{3} \left( \frac{x(x+1)(x-2)}{(x^2+1)(2x+3)} \right) \left( \frac{1}{x} + \frac{1}{x+1} + \frac{1}{x-2} - \frac{2x}{x^2+1} - \frac{2}{2x+3} \right).
\]

\(^1,2\) L'Hôpital’s rule
PROBLEM 2. Evaluate the integral
\[ \int \frac{\sec x \, dx}{\sqrt{\ln(\sec x + \tan x)}} \]

SOLUTION. Let \( u = \ln(\sec x + \tan x) \). Then
\[ du = \frac{\sec^2 x + \tan x \sec x}{\sec x + \tan x} \, dx = \sec x \, dx \quad \text{and} \quad \int = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{\ln(\sec x + \tan x)} + C. \]

PROBLEM 3. Find \( dy/d\theta \), where \( y = \theta^3 e^{-2\theta} \cos 5\theta \).

SOLUTION. Just the product rule and chain rule:
\[ y' = 3\theta^2 e^{-2\theta} \cos 5\theta + \theta^3 \cdot (-2)e^{-2\theta} \cos 5\theta + \theta^3 e^{-2\theta} \cdot (-5 \sin 5\theta). \]

Assignment 5 (due 22/04 by 9 am): page 378, no. 4; page 386, no. 26; page 392, no. 24(a).

PROBLEM 1. The solid lies between planes perpendicular to the \( x \)-axis at \( x = -1 \) and \( x = 1 \). The cross sections perpendicular to the \( x \)-axis are circular disks whose diameters run from the parabola \( y = x^2 \) to the parabola \( y = 2 - x^2 \). Find the volume of the solid.

SOLUTION. The diameter of the disk at position \( x \) is \( d(x) = |x^2 - (2 - x^2)| = |2x^2 - 2|; \) hence, its area is \( A(x) = \frac{1}{2} \pi [d(x)]^2 = \pi (x^2 - 1)^2 \), and the volume is
\[ V = \int_{-1}^{1} A(x) \, dx = \int_{-1}^{1} \pi (x^2 - 1)^2 \, dx = \int_{-1}^{1} \pi (x^4 - 2x^2 + 1) \, dx = \pi \left( \frac{x^5}{5} - \frac{2x^3}{3} + x \right) \bigg|_{-1}^{1} = \frac{16\pi}{15}. \]

PROBLEM 2. Find the volume of the solid generated by revolving the region bounded by the curve \( y = 4 - x^2 \) and the line \( y = 2 - x \) about the \( x \)-axis.

SOLUTION. Find the points of intersection of the curves: \( 4 - x^2 = 2 - x \) gives us \( x = -1 \) or \( x = 2 \). These are the limits of integration. If \( x \in [-1, 2] \), then \( 4 - x^2 \geq 2 - x \). Thus, using the disk method, the volume is
\[ V = \int_{-1}^{2} \pi \left[ (4 - x^2)^2 - (2 - x)^2 \right] \, dx = \int_{-1}^{2} \pi (x^4 - 9x^2 + 12x) \, dx = \frac{108\pi}{5}. \]

PROBLEM 3. Use the shell method to find the volume of the solid generated by revolving the region bounded by the curves \( x = \frac{y^4}{4} - \frac{y^2}{2} \) and \( x = \frac{y^2}{2} \) about the \( x \)-axis.

SOLUTION. Find the intersection points: \( \frac{y^4}{4} - \frac{y^2}{2} = \frac{y^2}{2} \) gives us \( y = 0 \) or \( y = \pm 2 \). Note that the region is symmetric against the axis of revolution (= \( x \)-axis), and to find the volume we should consider only one half, say, the one between \( y = 0 \) and \( y = 2 \). These are the limits of integration. For the shell method and revolution about the \( x \)-axis we should integrate with respect to \( y \), and the limits are \( -2 \) and \( 2 \). On the segment \( y \in [0, 2] \) we have \( \frac{y^4}{4} - \frac{y^2}{2} \leq \frac{y^2}{2} \). Apply the formula to get
\[ V = \int_{0}^{2} 2\pi y \left( \frac{y^2}{2} - \left( \frac{y^4}{4} - \frac{y^2}{2} \right) \right) \, dy = \int_{0}^{2} 2\pi \left( y^3 - \frac{y^5}{4} \right) \, dy = \frac{8\pi}{3}. \]

Assignment 4 (due 15/04 by 9 am): page 339, no. 52; page 344, no. 16; page 372, no. 20.

PROBLEM 1. Find \( dy/dx \), where \( y = \int_{0}^{x^2} \cos \sqrt{t} \, dt \).

SOLUTION. By the chain rule:
\[ \frac{d}{dx} \int_{0}^{x^2} \cos \sqrt{t} \, dt = \cos \sqrt{x^2} \cdot \frac{d(x^2)}{dx} = 2x \cos x. \]
Remark: strictly speaking, we should have put $\cos|x|$ in the result. However, since $\cos(-x) = \cos x$, the absolute value is redundant here. □

**Problem 2.** Evaluate $\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2}$.

**Solution.** Let $u = 1 + \sqrt{y}$. Then $du = dy/2\sqrt{y}$ and

\[
\int_1^4 \frac{dy}{2\sqrt{y}(1+\sqrt{y})^2} = \int_{1+\sqrt{1}}^{1+\sqrt{3}} \frac{1}{u^2} \, du = -\left[ \frac{1}{u} \right]_1^\sqrt{3} = \frac{1}{2}.
\]

**Problem 3.** Find the area of the region enclosed by the curve $y = x\sqrt{a^2 - x^2}$, $a > 0$, and the line $y = 0$.

**Solution.** Sketch the curve to see that it intersects the line $y = 0$ (i.e., the $x$-axis) at three points, $x = \pm a$ and $x = 0$, and it lies below (resp., above) the $x$-axis when $x \in [-a,0]$ (resp., $x \in [0,a]$). Thus, the area is

\[
- \int_{-a}^0 x\sqrt{a^2 - x^2} \, dx + \int_0^a x\sqrt{a^2 - x^2} \, dx = 2 \int_0^a x\sqrt{a^2 - x^2} \, dx = - \int_{a^2}^{a^2} \sqrt{u} \, du = \frac{2\sqrt{a^3}}{3} = \frac{2a^3}{3}.
\]

Remark: the two integrals are equal, up to sign, via the change of variables $u = -x$, or, equivalently, due to the symmetry of the curve. □

**Assignment 3 (due 08/04 by 9 am):** page 244, no. 26; page 245, no. 32(a).

**Problem 1.** A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per a square unit of surface area is twice as big for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be kept to a minimum. Neglect the thickness of the silo and the waste in construction.

(Hint: when a problem is stated using the terms ‘proportional’, ‘twice as big as’, ‘is fixed’, etc., i.e., particular coefficients/values are not given, denote them by letters and hope that they will not affect the final result.)

**Solution.** Let $V = \text{const}$ be the volume of the silo, and let $h$ and $r$ be the height of its cylindrical sidewall and the radius of its base, respectively. Denote by $c$ the cost of construction (per square unit of surface area) of the cylindrical part. Then the volume of the silo is $V = \pi r^2 h + \frac{4}{3} \pi r^3$, and the cost of its construction is $C(r,h) = c \cdot 2 \pi r h + 2c \cdot 2 \pi r^2$. Find $h$ from the former equation and plug it into the latter:

\[
h = \frac{V}{2\pi r^2} - \frac{2}{3} r,
\]

\[
C(r) = 2c \left( \frac{V}{r} + \frac{4\pi}{3} r^2 \right).
\]

The bounds for $r$ are found from the conditions $r > 0$ and $h \geq 0$; the latter gives $r \leq \sqrt{\frac{3V}{2\pi}}$. (Note the $\geq$ in this inequality: the silo may have no cylindrical part at all!) Thus, we should minimize $C(r)$ above on the semi-open interval $(0, \sqrt{\frac{3V}{2\pi}}]$. Differentiate and find critical points: $C'(r) = 2c (8 \pi r^3 - 3V)/3r^2$, and the only critical point on the interval is $r = \sqrt{\frac{3V}{8\pi}}$. Checking the sign of the derivative shows that this is a local minimum; hence, the function attains a global minimum here.

To write down the result (as we do not know $V$), we can express $V$, and then $h$, in terms of $r$. We get $V = 8 \pi r^3/3$ and $h = 2r$. Thus, the cost is minimal when the dimensions are related as $h/r = 2$. □

**Problem 2.** The stiffness $S$ of a rectangular beam is proportional to its width times the cube of its depth. Find the dimensions of the stiffest beam that can be cut from a 12-in.-diameter log.

**Solution.** Let $w$ be the width and $d$, the depth. Then $S = kwd^3$ (for some constant $k$) and we have $w^2 + d^2 = (12)^2$ (as the rectangular is inscribed in a disk of diameter 12). Resolve the latter equation in $w$, $w = \sqrt{(12)^2 - d^2}$, and plug the result into the former: $S(d) = k d^3 \sqrt{(12)^2 - d^2}$. This function is to be maximized on $d \in (0,12)$. Differentiate, $S'(d) = 4kd(3d^2 - d^2)/\sqrt{(12)^2 - d^2}$, and find the critical points: $d = 0$, $\pm 12$, and $\pm 6\sqrt{3}$. Only one critical point, $d = 6\sqrt{3}$, is in the interval in question. The derivative is positive on $(0,6\sqrt{3})$, and it is negative on $(6\sqrt{3},12)$. Hence, the point found is a point of local maximum, and since it’s the only critical point, it is also a point of global maximum. Finally, the beam of maximal stiffness has dimensions $w = 6$ in (width) and $d = 6\sqrt{3}$ in (depth). □

**Assignment 2 (due 01/04 in class):** page 208, no. 22(a–c); page 231, no. 54; page 232, no. 78.
Problem 1. Let \( g(x) = \sqrt{x} - 5 - x \). (a) Find the intervals on which the function is increasing and decreasing. (b) Then identify the function’s local extreme values, if any, saying where they are taken on. (c) Which, if any, of the extreme values are absolute?

Solution. First of all, observe that the function is only defined for \( x \leq 5 \), i.e., on \((-\infty, 5]\). Find the first derivative: \( g'(x) = -5x(x-4)/2\sqrt{5-x} \). It vanishes at \( x = 0 \) and \( x = 4 \) and is undefined at \( x = 5 \). Thus, \( x = 0, 4, \) and \( 5 \) are the critical points. The derivative is negative at \( (0, 4) \), and it changes sign at 0 and 4. (Note that we cannot start from \( +\infty \) here!) Hence, (a) the function is decreasing on \((-\infty, 0) \) and \((4, 5]\), and it is increasing on \((0, 4) \). (b) The points 0 and 5 are local minima (with \( g(0) = g(5) = 0 \)), and \( x = 4 \) is a local maximum (with \( g(4) = 16 \)). (c) The minima \( x = 0 \) and \( x = 5 \) are absolute; the maximum \( x = 4 \) is not. To see this, one can sketch the graph. For the latter assertion we should also make sure that the function has values larger than 16. (Note that this is not obvious from the graph, as we don’t know the behavior of the branch going to \(-\infty\).) This follows, e.g., from the fact that \( \lim_{x \to -\infty} g(x) = \infty \). □

Problem 2. Graph the rational function

\[
y = \frac{x^2 - 4}{x^2 - 2}.
\]

Include the graphs and equations of the asymptotes and dominant terms.

Solution. We have \( y' = \frac{4x}{(x^2 - 2)^2} \) and \( y'' = \frac{-4(3x^2 + 2)(x^2 - 2)^3}{x^2 - 2} \). The first derivative vanishes at \( x = 0 \), the second derivative has no roots. Both are undefined at \( x = \pm \sqrt{2} \) (which are critical points, but we need to consider them anyway, as the derivatives may change sign there). Here are the signs:

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-\sqrt{2})</th>
<th>0</th>
<th>(\sqrt{2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y')</td>
<td>-</td>
<td>0</td>
<td>+</td>
</tr>
<tr>
<td>(y'')</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>(y)</td>
<td>(\infty)</td>
<td>min</td>
<td>(\infty)</td>
</tr>
</tbody>
</table>

Remark: note that you cannot claim that the function is, say, decreasing on \((-\infty, 0) \), as it is undefined in the middle! For the same reason, \( \pm \sqrt{2} \) are not inflection points.

The graph has two vertical asymptotes \( x = \pm \sqrt{2} \) (the points where \( \lim y = \infty \)) and a two-sided horizontal asymptote \( y = 1 \) (as \( \lim_{x \to \infty} y = 1 \)). I cannot draw a graph here, but it’s pretty obvious now :) □

Problem 3. (Good ol’ limits, just to keep you alert.) Find \( \lim_{x \to \infty} (\sqrt{x^2 + x} - \sqrt{x^2 - x}) \).

Solution.

\[
\lim_{x \to \infty} (\sqrt{x^2 + x} - \sqrt{x^2 - x}) = \lim_{x \to \infty} \frac{(x^2 + x) - (x^2 - x)}{\sqrt{x^2 + x} + \sqrt{x^2 - x}} = \lim_{x \to \infty} \frac{2}{\sqrt{1 + 1/x} + \sqrt{1 - 1/x}} = 1. \]

Assignment 1 (due 25/03): page 195, no. 16; page 217, no. 18; page 269, no. 4.

Problem 1. Find the absolute maximum and minimum values of the function \( g(x) = -\sqrt{5} - x^2 \) on the interval \(-\sqrt{5} \leq x \leq 0 \). Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

Solution. \( g'(x) = x/\sqrt{5 - x^2} \). The numerator of this expression vanishes at \( x = 0 \), the denominator, at \( x = \pm \sqrt{5} \). Hence, there are three critical points, 0 and \( \pm \sqrt{5} \), but only two of them, 0 and \( -\sqrt{5} \), belong to the interval in question. We should add to this list the endpoints, but they are there already. Comparing \( g(0) = -\sqrt{5} \) and \( g(-\sqrt{5}) = 0 \), we conclude that \( g \) attains an absolute maximum of 0 at \( x = -\sqrt{5} \) and an absolute minimum of \(-\sqrt{5} \) at \( x = 0 \). The graph is a parabola of the 3rd quadrant. □

Problem 2. Use the steps of the graphing procedure on page 214\(^1\) to graph the equation \( y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4 \). Include the coordinates of any local extreme points and inflection points.

Solution. \( g'(x) = \frac{\sqrt{3}}{2} \). The equation \( y' = 0 \) has solutions \( x = -\sqrt{3} \), \( x = 0 \), and \( x = \sqrt{3} \), and the signs of \( y' \) on the four resulting intervals are +, −, +, −. Hence, the function is increasing on \((-\infty, -\sqrt{3}) \) and \((0, \sqrt{3}) \), and it is decreasing on \((\sqrt{3}, 0) \) and \((\sqrt{3}, +\infty) \). The points \( \pm \sqrt{3} \) are local maxima, and 0 is a local minimum.

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\(^1\)Strategy for Graphing \( y = f(x) \): (1) Find \( y' \) and \( y'' \). (2) Find the rise and fall of the curve. (3) Determine the concavity of the curve. (4) Make a summary and show the curve’s general shape. (5) Plot specific points and sketch the curve.
Step 3. The equation $y'' = 0$ has solutions $x = \pm 1$, and the signs of $y''$ on the three resulting intervals are $-, +, -$. Hence, the graph is concave down on $(-\infty, -1)$ and $(1, \infty)$, and it is concave up on $(-1, 1)$; the points $\pm 1$ are points of inflexion.

Step 4. Sorry, .pdf does not seem to accept .eps graphics... But here it is straightforward.

Step 5. The specific points of the graph are: two maxima $(\pm \sqrt{3}, 5)$, a minimum $(0, -3)$, and two inflection points $(\pm 1, 1)$. These points are to be plotted and connected with a ‘nice smooth’ curve. □

Problem 3. Find values of $a$ and $b$ such that the function

$$f(x) = \frac{ax + b}{x^2 - 1}$$

has a local extreme value of 1 at $x = 3$. Is this extreme value a local maximum, or a local minimum? Give reasons for your answer.

Solution. Find $f'(x) = \frac{[a(x^2 - 1) - 2x(ax + b)]/(x^2 - 1)^2}. We want the function to have a critical point at $x = 3$. The derivative is defined at $x = 3$, hence, it should vanish there. Thus, one must have $a(3^2 - 1) - 2(3)(a(3) + b) = 0$, which amounts to $10a + 6b = 0$. Call this (*). Furthermore, we must have $f(3) = 1$, which gives $(3a + b)/8 = 1 (**). Solving (*) and (**) together gives $a = 6, b = -10$.

Thus, the function is $f(x) = (6x - 10)/(x^2 - 1)$, and its first derivative is $f'(x) = (-6x^2 + 20x - 6)/(x^2 - 1) = (-6x + 2)(x - 3)/(x^2 - 1)$. The latter expression changes sign from $+$ to $-$ at $x = 3$. Hence, $x = 3$ is a point of local maximum. □