Problem of the Month

March 2015

Problem:
In each step one can choose two indices $1 \leq k, l \leq 100$ and transform the 100 tuple \((a_1, \ldots, a_k, \ldots, a_l, \ldots, a_{100})\) into the 100 tuple \((a_1, \ldots, \frac{a_k}{2}, \ldots, a_l + \frac{a_k}{2}, \ldots, a_{100})\) if \(a_k\) is an even number. We say that a permutation \((a_1, \ldots, a_{100})\) of \((1, 2, \ldots, 100)\) is good if starting from \((1, 2, \ldots, 100)\) one can obtain it after finite number of steps. Find the total number of distinct good permutations of \((1, 2, \ldots, 100)\).

Solution: The answer is 100! By the method of mathematical induction we prove that starting from \((1, 2, \ldots, n)\) we can reach any permutation of \((1, 2, \ldots, n)\) for any nonnegative value of \(n\).

1. The case \(n = 2\) is clear.
2. Suppose that the statement is true for \(n = r - 1\). We will show that starting from \((1, 2, \ldots, r)\) we can reach its arbitrary permutation \((a_1, a_2, \ldots, a_r)\). Let \(s\) be an index such that \(a_s = r\). First of all, we are going to place \(r\) into his desired place by proving that we can reach the permutation

\[
(r - s + 1, r - s + 2, \ldots, r - 2, r - 1, r, 1, 2, \ldots, r - s - 1, r - s) \quad (1)
\]

Let \(T(l)\) be the transformation when the half of the entry \(a_l\) has added to \(a_{l+1}\) \((a_{r+1} \equiv a_1)\):

\[
(a_1, \ldots, a_{l-1}, a_l, a_{l+1}, \ldots, a_k) \rightarrow (a_1, \ldots, a_{l-1}, \frac{a_l}{2}, a_{l+1} + \frac{a_l}{2}, \ldots, a_r)
\]

It can be readily seen that the series of transformations \(T(2), T(3), \ldots, T(r)\) is a cyclic transformation: it shifts \((1, 2, \ldots, r)\) to \((r, 1, 2, \ldots, r-1)\). Similarly, the series \(T(3), T(4), \ldots, T(r), T(1)\) will shift \((r, 1, 2, \ldots, r-1)\) to \((r-1, r, 1, 2, \ldots, r-2)\) and \(T(4), T(5), \ldots, T(r), T(1)\), \(T(2)\) will shift \((r-1, r, 1, 2, \ldots, r-2)\) to \((r-2, r-1, r, 1, 2, \ldots, r-3)\). Thus, after \(s\) similar shifts we will get the desired permutation \((1)\). Now the entry \(r\) is correctly located, the set of remaining entries is \(\{1, 2, \ldots, r - 1\}\) and by inductive hypothesis we can get the desired permutation \((a_1, a_2, \ldots, a_r)\). Done.