Problem Of The Month

February 2015

Problem:

Is there a set of 2015 consecutive positive integers containing exactly 15 prime numbers?

Solution: The answer is yes.

For each positive integer $n$ let $f(n)$ be the number of prime numbers among $n, n+1, \ldots, n+2014$. We will show that $f(k) = 15$ for some positive integer number $k$. First of all we note that

- $f(1) \geq 15$ since $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47$ are prime numbers.
- $f(2016!+2) = 0$ since for each $2 \leq l \leq 2016$ the number $2016!+l$ is not a prime number.

Now note that by the definition for each positive $n$ the difference $f(n+1) - f(n)$ is equal to $0, -1$ or $1$. In other words, while $n$ increases by $1$, $f(n)$ can change only by $1$. Thus, when $n$ changes from $1$ to $2016! + 2$, $f(n)$ smoothly (at most by $1$) changes from some number exceeding $15$ to $0$. Therefore, for some integer $1 < k < 2016! + 2$ we have $f(k) = 15$. Done.