Problem:

Show that there is a positive integer \( p \) for which there exists a sequence of positive integers \( \{x_n\}_{n=1}^{\infty} \) such that

- each \( x_n \) is a sum of at most \( p \) powers of 2: \( x_n = 2^{l_1} + 2^{l_2} + \cdots + 2^{l_k} \), where \( k \leq p \)

and

- each \( x_n \) is divisible by \( 10^n \).

What is the minimal possible value of \( p \)?

Solution:

The answer: \( p = 2 \). First of all, since the power of 2 is not a multiple of 10, \( p = 1 \) does not satisfy the conditions. Now let us show that the sequence

\[
x_n = 2^n + 2^{2 \cdot 5^{n-1} + n}, \quad n = 1, 2, \ldots
\]

meets the conditions. Since \( x_n = 2^n(2^{2 \cdot 5^{n-1}} + 1) \) by we prove by induction that \( 2^{2 \cdot 5^{n-1}} + 1 \) is divisible by \( 5^n \).

\( \circ \) \( n = 1 : 2^2 + 1 \) is divisible by \( 5^1 \).

\( \circ \circ \) Suppose that \( 2^{2 \cdot 5^{k-1}} + 1 = 4^{5^{k-1}} + 1 \) is divisible by \( 5^k \). We have to show that \( 2^{2 \cdot 5^k} + 1 \) is divisible by \( 5^{k+1} \). Let \( t = 4^{5^{k-1}} \). Then \( 2^{2 \cdot 5^k} + 1 = t^5 + 1 = (t + 1)(t^4 - t^3 + t^2 - t + 1) \). By inductive hypothesis the first factor \( t + 1 \) is divisible by \( 5^k \). Thus, we have to show that \( t^4 - t^3 + t^2 - t + 1 \) is divisible by 5. Since \( t + 1 \) is divisible by 5, \( t = 5s - 1 \) and \( t^4 = t^2 = 1 \) and \( t^3 = t = -1 \) in mod 5. Therefore, \( t^4 - t^3 + t^2 - t + 1 \) is divisible by 5.

The proof is completed.