Problem Of The Month

June 2013

Problem:

Determine all positive integers \( n \) for which \( \frac{n! - 1}{2n + 7} \) is also an integer number.

Solution: The answer: \( n = 1, 5, 8 \).

It can be readily checked out that among first 6 integers 1 and 5 are only integers for which \( \frac{n! - 1}{2n + 7} \) is also integer.

Let \( n \geq 7 \) be an integer for which \( \frac{n! - 1}{2n + 7} \) is also an integer. Then if \( 2n + 7 \) is not prime then it has a prime divisor \( p_1 \leq n \). Contradiction, since \( p_1 \) also divides \( n! \). Thus, \( 2n + 7 = p \) is a prime number. We get

\[
(p-7)! \equiv 1 \pmod{p}
\]

Now by Wilson theorem \( (p-1)! \equiv -1 \pmod{p} \) and also

\[
(p-1)! \equiv (-1)^{\frac{p-1}{2}}(\left(\frac{p-1}{2}\right)!)^2 \cdot \frac{p-5}{2} \cdot \frac{p-3}{2} \cdot \frac{p-1}{2} \cdot \frac{p+1}{2} \cdot \frac{p+3}{2} \cdot \frac{p+1}{2} \equiv (-1)^{\frac{p-1}{2}} \frac{225}{64} \pmod{p}
\]

Therefore, \( 225 \equiv (-1)^{\frac{p+1}{2}} \pmod{p} \). If \( p = 4l + 1 \) then \( p|225 + 64 = 17^2 \) but \( p \geq 21 \), no solution. If \( p = 4l + 3 \) then \( p|225 - 64 = 7 \cdot 23 \). Since \( p \geq 21 \) we get \( p = 23 \) and \( n = 8 \) which satisfies the condition. Done.