Bilkent University  
Department of Mathematics  

Problem Of The Month  

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Problem:  

Some unit squares of the grid $99 \times 99$ are marked so that any sub-square $5 \times 5$ of the grid consisting of unit squares has at least 6 marked unit squares. What is the minimal possible number of marked unit squares?  

Solution: The answer is 2261.  

Suppose that the centers of unit squares have coordinates $(i, j)$, where $i = 1, 2, \ldots, 99; j = 1, 2, \ldots, 99$. The unit square with center at $(i, j)$ will be denoted by $u(i, j)$. Let the marked unit squares are:  

$u(5k, 5l + 1)$, where $1 \leq k \leq 19, 0 \leq l \leq 19$ and  

$u(m, 5n)$, where $1 \leq m \leq 99, 1 \leq n \leq 19$.  

Then it can be readily seen that the total number of marked unit squares is 2261, and any sub-square $5 \times 5$ has exactly 6 marked unit squares.  

Let $k$ be a positive integer. Now by the method of mathematical induction we’ll show that if any $5 \times 5$ sub-square of the grid $(5k + 4) \times (5k + 4)$ has at least 6 marked unit squares, then the total number of marked unit squares is at least $6k^2 + 5k$.  

• $k = 1$. $6 \cdot 1^2 + 5 \cdot 1 = 11$. Consider two $5 \times 5$ squares: the square consisting all $u(k, l)$, where $1 \leq k \leq 5, 1 \leq l \leq 5$ and the square consisting all $u(k, l)$, where $5 \leq k \leq 9, 5 \leq l \leq 9$. Each of these $5 \times 5$ squares contains at least 6 marked unit squares and their intersection
is the unit square \(u(5,5)\). Therefore the total number of marked unit squares is at least 11. Done.

- Suppose the statement is correct for a \((5k + 4) \times (5k + 4)\) grid \(A\) and consider a \((5k + 9) \times (5k + 9)\) grid \(B\). Suppose that \(A\) consists of all unit squares \(u(i,j)\), where \(1 \leq i \leq 5k + 4, 1 \leq j \leq 5k + 4\) and \(B\) consisting of all unit squares \(u(i,j)\), where \(1 \leq i \leq 5k + 9, 1 \leq j \leq 5k + 9\).

Let \(5 \times 5\) squares \(U_s, s = 1,2,\ldots,k+1\); consist of all unit squares \(u(i,j)\), where \(5k + 5 \leq i \leq 5k + 9, 5s - 4 \leq j \leq 5s\) and \(5 \times 5\) squares \(V_t, t = 1,2,\ldots,k+1\); consist of all unit squares \(u(i,j)\), where \(5t - 4 \leq i \leq 5t, 5k + 5 \leq j \leq 5k + 9\). Note that the squares \(U_{k+1}\) and \(V_{k+1}\) share a unit square \(u(5k + 5, 5k + 5)\), all other pairs of \(U_s\) and \(V_t\) squares do not share any unit square. Therefore, since the union of \(k + 1\) \(U_s\) and \(k + 1\) \(V_t\) squares is a subset of the set \(B - A\), the set \(B - A\) contains at least \(6 \cdot 2(k + 1) - 1 = 12k + 11\) marked squares. Thus, by inductive hypothesis \(B\) contains at least \(6k^2 + 5k + 12k + 11 = 6(k + 1)^2 + 5(k + 1)\). Done.

At \(k = 19\) we get that the grid \(99 \times 99\) contains at least 2261 marked unit squares. The solution is completed.