Problem:

Find the maximal possible value of the real number $T$ such that for all positive real numbers $a, b, c$ satisfying $abc = 1$ we have

$$\frac{a + b}{ab + a + b} + \frac{b + c}{bc + b + c} + \frac{c + a}{ca + c + a} \geq T$$

Solution:

Let us show that

$$\frac{a + b}{ab + a + b} + \frac{b + c}{bc + b + c} + \frac{c + a}{ca + c + a} \geq 2 \quad \dagger$$

The substitution $a = x^3, b = y^3, c = z^3$ yields:

$$\frac{x^3 + y^3}{x^3y^3 + x^3 + y^3} + \frac{y^3 + z^3}{y^3z^3 + y^3 + z^3} + \frac{z^3 + x^3}{z^3x^3 + z^3 + x^3} \geq 2$$

Let us prove that

$$\frac{x^3 + y^3}{x^3y^3 + x^3 + y^3} \geq \frac{xz + yz}{xy + yz + xz} \quad \ddagger$$
Since $x, y, z$ are positive, the inequality (‡) is equivalent to $(x^3 + y^3)(xy + yz + xz) \geq (xz + yz)(x^3y^3 + x^3 + y^3)$ or $x^3 + y^3 \geq x^3y^2z + x^2y^3z$. The last inequality holds since $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$ and $x^3y^2z + x^2y^3z = x^2y + xy^2 = (x + y)xy$. The inequality (‡) is proved. The similar inequalities can be obtained for $y, z$ and $z, x$. The sum of these three inequalities yields (†). $T = 2$ is achieved at $a = b = c = 1$. Done.