Problem Of The Month

November 2009

Problem:

Suppose that the set of all natural numbers \( N \) is partitioned into 3 pairwise disjoint infinite sets \( A, B \) and \( C \): \( A \cup B \cup C = N \). Prove that there are infinitely many triples \( a \in A, b \in B \) and \( c \in C \) such that \( a, b \) and \( c \) are sides of some triangle.

Solution:

Assume that there are only finitely many triples \( a \in A, b \in B \) and \( c \in C \) such that \( a, b \) and \( c \) are sides of some triangle. Since the sets \( A, B \) and \( C \) are infinite, there exist natural numbers \( a_1 \in A, b_1 \in B \) and \( c_1 \in C \) exceeding all these triangle sides and satisfying \( 1 < a_1 < b_1 < c_1 \). Obviously, there is a triangle with sides \( a_1, c_1, c_1 + 1 \), as well as a triangle with sides \( b_1, c_1, c_1 + 1 \). Therefore, by assumption, \( c_1 + 1 \in C \). By repeating this argument, we get that all natural numbers exceeding \( c_1 \) belong to \( C \), and as a consequence the sets \( A \) and \( B \) are finite. A contradiction.