Problem Of The Month

June 2009

Problem:

Find all natural numbers $a$ and $b$ such that $a \neq b$ and for some prime $p$ and natural numbers $k, n$

$$b^2 + a = p^k \text{ and } a^2 + b = np^k$$

Solution:

Suppose that $b = 1$. Then $a + 1$ divides $a^2 + 1$. Therefore, $a + 1$ divides 2 and $a = 1$. Contradiction with $a \neq b$.

Suppose that $b > 1$ and $b^2 + a = p^k$. Then $a^2 + b \equiv 0 \mod (b^2 + a)$ and $b^2 + a \equiv 0 \mod (b^2 + a)$. Therefore, in $\mod (b^2 + a)$ we have $b = -a^2$ and $b^4 = a^2$. As a result, $b^2 + a$ divides $b^4 + b$. Since $b^4 + b = b(b^3 + 1)$, $\gcd(b, b^3 + 1) = 1$ and $b^2 + a = p^k$

there are two possibilities: $b^2 + a$ divides $b$ or $b^2 + a$ divides $b^3 + 1$. The first case is impossible. Thus, $b^2 + a$ divides $b^3 + 1 = (b + 1)(b^2 - b + 1)$. Now note that both factors are not divisible by $b^2 + a$, since $b^2 + a$ and $b^2 - b + 1 < b^2$. Therefore, both factors are divisible by $p$, since $b^2 + a = p^k$. Thus, $p$ divides $\gcd(b + 1, b^2 - b + 1)$.

Since $b^2 - b + 1 \equiv 3 \mod (b + 1)$, we get $p = 3$. As a result, $3^k$ divides $(b + 1)(b^2 - b + 1)$.

$k \neq 1$. If $k = 2$, then $b = 2$ and $a = 5$. Suppose that $k \geq 3$. Easy check shows that $b^2 - b + 1$ is not divisible by 9. Therefore, $3^{k-1}$ divides $b + 1$. But then $b \geq 3^{k-1} - 1$ or $b^2 \geq (3^{k-1} - 1)^2$. Finally, $3^{k-1} = \frac{b^2 + a}{3} > \frac{(3^{k-1} - 1)^2}{3} > 3^{k-1}$ for $k \geq 3$.

Contradiction. The only solution is $a = 5$, $b = 2$. 