Problem Of The Month

March 2009

Problem:

Let \( x_i, i = 1, 2, \ldots, 2009 \) be real numbers satisfying

\[
\sum_{i=1}^{2009} \frac{1}{x_i^2 + 1} = 2008.
\]

Find the maximum of the expression \( \sum_{(i,j)} x_ix_j \), where the summation is taken over all pairs \( (i, j) : i, j = 1, 2, \ldots, 2009; i > j \).

Solution:

The answer is \( \frac{2009}{2} \).

First of all, we note that

\[
\sum_{i=1}^{2009} \frac{x_i^2}{x_i^2 + 1} = \sum_{i=1}^{2009} \left(1 - \frac{1}{x_i^2 + 1}\right) = 2009 - 2008 = 1.
\]

Now by Cauchy- Schwarz inequality

\[
\left( \sum_{i=1}^{2009} \frac{x_i^2}{x_i^2 + 1} \right) \left( \sum_{i=1}^{2009} (x_i^2 + 1) \right) \geq \left( \sum_{i=1}^{2009} \frac{x_i}{\sqrt{x_i^2 + 1}} \cdot \sqrt{x_i^2 + 1} \right)^2 = \left( \sum_{i=1}^{2009} x_i \right)^2
\]
and therefore, $\sum_{i=1}^{2009} (x_i^2 + 1) \geq \left( \sum_{i=1}^{2009} x_i \right)^2$ or $\sum_{i,j} x_i x_j \leq \frac{2009}{2}$. The equality holds when $x_1 = x_2 = \cdots = x_{2009} = \sqrt{\frac{1}{2008}}$. 