Instructions:

- Attempt at most TWO questions from each of the four sections.
- Hand in four separate scripts, one script for each section.
- Write your name and the section letter on each sheet that you hand in.

Failure to follow the instructions will cause inconvenience and may cause marking omissions.

Time allowed: three hours.
Section A: Algebra

(1) Let \( p \) be a prime. How many isomorphism classes of groups \( G \) are there such that \(|G| = p^3\) and every non-identity element of \( G \) has order \( p \)?

(2) (a) Give an example of a field \( K \) such that \( \text{Gal}(K/\mathbb{Q}) \cong C_6 \). How many strictly intermediate fields \( \mathbb{Q} < I < K \) are there?

(b) Give an example of a field \( L \) such that \( \text{Gal}(L/\mathbb{Q}) \cong S_6 \). How many strictly intermediate fields \( \mathbb{Q} < J < L \) are there?

(3) Assuming the existence of the algebraic closure of a given field, prove the uniqueness.

(4) Construct the ordinary character table for the group \( A_4 \times C_2 \).
Section B: Analysis

(1) (a) Let $f$ be a holomorphic in $D = \{ z \in \mathbb{C} : |z| < 1 \}$ function such that $f'(1/n) = f(1/n)$ for $n = 2, 3, \ldots$. Show that $f$ is an entire function.

(b) Is there a non-zero holomorphic in $D$ function $f$ with $|f(1/n)| < 2^{-n}$ for all $n \geq 2$?

(2) Determine the largest open set to which the function $f(z) = \sum_{1}^{\infty} z^n/n^2$ can be analytically continued.

(3) Determine which of the following statements are true or false. If false, give a counter-example.

(a) If $f_n \rightarrow f$ in $L_2(X, \mu)$ then $f_n \overset{\mu}{\rightarrow} f$.

(b) If $f_n \rightarrow f$ in $L_1(\mathbb{R})$ and $f_n \rightarrow g$ in $L_2(\mathbb{R})$ then $f \equiv g$.

(c) If $f_n \overset{\text{au}}{\rightarrow} f$ and $|f| \leq g$ and $g \in L_2(X, \mu)$ then $f_n \rightarrow f$ in $L_1(X, \mu)$.

(d) If $f_n \overset{\mu}{\rightarrow} f$ and $\lambda << \mu$ then $f_n \overset{\lambda}{\rightarrow} f$.

(4) Show that the unit sphere in a Hilbert space is not weakly compact.
Section C: Applied Mathematics

(1) Use the method of Frobenius to obtain the general solution to the differential equation
$$z^2 u'' - 2zu' + (2 - z^2)u = 0,$$
valid near $z = 0$.

(2) Recall that the Hermite polynomials $H_n$ have generating function
$$e^{-t^2+2xt} = \sum_{n=0}^{\infty} H_n(x)t^n/n!.$$
Show that:
(a) $H_n(-x) = (-1)^n H_n(x)$,
(b) $H_{n+1} - 2xH_n = 2nH_{n-1}$,
(c) $H_n'' - 2xH_n' + 2nH_n = 0$.

(3) Find the formal solution to $u_t = au_{xx}$ where $a > 0$, $t > 0$, $x \in [0, L]$ with $u(0, t) = u(L, t)$. Express the solutions in terms of the constants $\alpha_n$ where
$$u(x, 0) = \sum_{n=1}^{\infty} \alpha_n \sin(n\pi x/L).$$
(You do not need to discuss the hypotheses under which the formal solution is valid.)

(4) Let $D$ be a bounded normal domain with boundary $B$. Consider the Dirichlet problem
$$\nabla^2 u = h \text{ in } D \quad \text{and} \quad u = f \text{ on } B$$
where $h$ and $f$ are continuous functions in $D$ and on $B$. Recall that the Green’s function $G(x, y)$ is given by
$$\nabla_y^2 G(x, y) = \delta(x - y) \text{ in } D \quad \text{and} \quad G(x, y) = 0 \text{ for } y \in B.$$
(a) Without proof, express $u$ in terms of $G$, $h$, $f$.
(b) Prove that $G(x, y) = G(y, x)$ for $x, y \in D$. 

Section D: Topology and Geometry

(There is no geometry in this paper)

(1) Show that, in any topological space $X$, each quasi-component is a union of components, and each component is a union of path-components. (Recall that the quasi-component of a point $x \in X$ is the intersection of all the open-closed subsets of $X$ containing $x$.)

(2) Let $K \subseteq \mathbb{R}^3$ be the trefoil knot (which can be thought of as the circle embedded in $\mathbb{R}^3$ via $f \mapsto (2 \cos 2f + \cos 3f, 2 \sin 2f + \cos 3f, \sin 3f)$ for $f \in [0, 2\pi]$; in other words, it wraps the standard torus two times along the parallels and three times along the meridians). Find the homology groups of the complement $\mathbb{R}^3 - K$. 