Lecture-classes: in Fen A, SAZ03, Tuesdays 13:40 - 14:30, Thursdays 15:40 - 17:30. If you miss a class, you must contact other students to find out the material covered in your absence, and you must work through that material on your own, preferably before attending any further classes. I do not wish to disrupt my narrative by giving recaps of material presented in previous classes. It is also your responsibility to be aware of all announcements that are made in class.

Office consultation hours: Fen A room 129, Tuesdays 16:40 - 17:30. If your homework or quizzes are inadequate, I may summon you to office hours by e-mail, giving you 24 hours notice, in which case attendance is compulsory unless a medical note can be produced. Although I may not be available at other times, this hour of my week belongs exclusively to the Algebra I course, and I strongly advise you to make use of it, especially if you are having difficulty with the course.

Warning: Group theory, the topic of Algebra I, is a “click topic”. At first, some of the concepts will seem rather difficult but, with experience and practise, the concepts will click, and the material will suddenly become easy. That presents some dilemmas for the teacher, who has to serve students at various different levels of proficiency.

The beginning of the course will proceed very slowly, and the introductory material will be accompanied by detailed written notes. That way, everyone will receive every possible opportunity to master the core parts of the course, and any stragglers can be deemed either too lazy or else intellectually too feeble for the mathematics undergraduate programme; and I shall be able to award any stragglers an F grade with a clear conscience. The quicker students are requested to attend those early classes patiently, and to be generous in recognizing the needs of the slower ones.

It takes time for some of the concepts to sink in. You cannot survive this course by muddling through for the first few weeks and then cramming just before Midterm I. To adequately prepare for the exams, you must do the homework every week. As a pandering to those who are lacking in self-discipline, I shall apply some enforcement: students who fail to make a reasonable effort in at least half of the homeworks or in at least half of the quizzes will receive a black mark, resulting in a reduction of their grade; see the section below on Course Assessment.

The teaching dilemmas are even more severe in Algebra II. It is impossible to communicate Galois theory to people who have not assimilated the basics of group theory. For that reason, failures in Algebra I will have to be properly labeled with an F, to disqualify them from continuation of the algebra programme.

If you master the Midterm I material, most of the Midterm II material and then fade during the few weeks before the Final, that would be satisfactory; I would consider you to have made an honorable effort, and you would be likely to receive a respectable grade.
**Source material:**

**Classroom lectures:** I shall be pursuing a particular narrative that does not follow along the lines of any textbook. Some parts of the narrative, especially the early parts, are to be accompanied by notes and handouts posted on my website, but I am under no obligation stick to my notes. If you miss a class, you must catch up on your own, extracting the material from textbooks and class notes made by other students.

Because you are a continuous assessment student, you may find it difficult to read books on your own, without a teacher to lead you through them. Some of the notation and terminology will be unfamiliar, and even the results that you already know will be stated differently, proved differently, and presented in a different order. But the ability to resolve different accounts of a topic is a basic academic skill. Each author has a unique personal conception of the material. If you wish to do well, you yourself must develop your own conception of the material.

**Required textbook:** W. Ledermann, A. J. Weir, *Introduction to group theory*, 2nd edition, Longman, 1996. This gives a leisurely account of the Algebra I material. Its contents are approximately the same as the Algebra I syllabus, though in a different style and with material presented in a quite different order. (During my school days, upon finding a copy of the first edition of this book in the school library, I worked my way through about half of the book, learning the theorems by writing out my own versions of the proofs. You have access to an improved and more detailed (and extended) version of the book, and you have access to much other literature besides. So, even if my teaching were to be so defective as to give you absolutely no help whatsoever, there can be only one possible possible excuse for failing to grasp the core material of the course, namely, inadequate ability for a university degree in mathematics.)


**Recommended textbook:** P. M. Cohn, *Basic algebra. Groups, rings and fields*, Springer-Verlag, 2003. This more advanced textbook gives a concise coverage of all the main material in Algebra I and Algebra II. It also introduces some deeper aspects of the foundations of algebra, particularly Zorn’s Lemma and the notion of a category.

M. Aschbacher, “Finite Group Theory”, Cambridge University Press, 1986. This is a more advanced text, aimed primarily at graduate students. Some of you might find the first few chapters useful as concise summaries of the definitions, theorems and proofs.

Many other texts on group theory can be found in the university library and on the internet. Wikipedia provides some lucid accounts of some standard mathematical definitions and well-known theorems.

**How to study for this course:** The educational system in which we operate, called *continuous assessment*, has some advantages and disadvantages in comparison to its exact opposite, called *independent study*. You are required to spend much of your time following paths laid out by your teachers. The main advantage of this system is that many of the students work quite hard. The main disadvantages are that (1) students tend to become focused on their external performance in assessments, rather than the internal development of their knowledge and understanding, (2) individuality is suppressed, erratic brilliance is undervalued, and students
may be misled into believing that quality can be measured by a single parameter.

I rather doubt that I, myself, could have survived a continuous assessment undergraduate programme. (I am lazy unless impelled by obsession, and too erratic to perform consistently in exams.) However, I cannot allow you the freedom to study mathematics in the way that I did when I was at your stage. I must force you to keep up with the lectures and the homeworks, otherwise most of you — very understandably, not your fault — would tend to give priority to the assessments that are constantly forced upon you in other courses. Alas, you and I have no choice but to submit to the system: I must force you to follow my path, not your own.

By means of the black mark system — see Course Assessment — I can compel you to produce acceptable homeworks. But there is something more vital which I cannot bring about though imposition of discipline, because it has to come from you. Take the time to do the homework slowly and patiently. The most effective learning takes place when you are struggling with a something that seems too difficult. You may spend half an hour thinking, producing nothing at the end of it. An unsuccessful effort to penetrate something too hard will certainly help you to clarify your understanding of the background concepts, and maybe you will make some other deductions and discoveries along the way. Then, when you turn to other exercises that also seemed too difficult, you will find that some of them have suddenly become easy. And, if you do eventually find out how to deal with some exercise that defeated you, the idea that you overlooked will mean something to you, and it will become etched into your mind.

The homework will not prepare you for the exams if you do it just with the aim of satisfying the assistant who marks them. You must do it to your own satisfaction. If you are not sure that you fully understand your own arguments, then you are in deep trouble, and you are in danger of eventually becoming a parrot. The parrots are the students who try to do mathematics by imitating form without regard to content. They search questions for familiar stimulus phrases and then reply with conditioned response phrases linked together by grammatically correct gibberish. The Algebra II course is always in danger of being infested by these creatures, which is why it is so important to exterminate those students who have already become parrots by the end of Algebra I.

Work through the lectured material. If an argument seems difficult, write out your own version of it. If you cannot follow some particular step, see if you can find some other way of filling in the gap. Learn the definitions, results, proofs, examples. Rote memorization may be better than nothing, but sometimes you will be required to adapt an idea to a different context; for that, you will need to understand what you are doing.

In finite group theory, the small examples are important. For each positive integer \( n \), there are only finitely many groups with order \( n \). And, for some small values of \( n \), you must be familiar with all of them. Later, in Algebra II, whenever you encounter a Galois extension of degree 8, you will consider the five groups with order 8 and, by considering their different properties, you will figure out which one is the Galois group.

I cannot take you there. It would be absurd if students were to come to class expecting some kind of winged teacher-monster to fly into the room and somehow magically beam knowledge and understanding into their brains. Teachers are mythical creatures, like unicorns. They do not exist. The verb \textit{to teach} is nonsensical, because it implies a passive form \textit{to be taught}, as if one could acquire knowledge and understanding without doing anything.

But, when you feel you do need help, you can dream that you are coming to my office. And, because you will already have thought about the matter in question, you will be able to dream of a discussion about it. Afterwards, when you have awoken and got back to your mathematics, you can assess the validity of the material that came up in the dream.
**Course Assessment:**

**Homeworks, 15%**. These are set every Thursday. You should complete a first draft of your homework before the Tuesday class. You may ask for help with the homework during the Tuesday class and the Tuesday office hours. Your completed homework is to be handed in on the Thursday. *Any students who fail to make a reasonable effort in at least 50% of the homeworks will be awarded a black mark.*

**Quizzes, 10%**. These will be set at a rate of at least one per week, sometimes without warning. In cases of low attendance (sometimes caused by imminent Midterms in other courses), I usually set very easy “attendance quizzes”. *Any students who are absent from more than 50% of the quizzes will receive a black mark.*

**Midterm I, 15%; Midterm II, 20%; Final, 40%**. Closed book proctored written exams. The Midterm I syllabus is a prerequisite for the Midterm II syllabus which, in turn, is a prerequisite for the Final syllabus. In the midterms, some of the questions will be very easy. I may ask for standard definitions, statements of standard results, proofs of standard results. Some questions may require adaptations of standard arguments. And sometimes I may set a more entertaining question just to see if there is anyone in the class who can do it. After each exam, you must come to my office — during office hours or during a special scheduled session — for an autopsy of your script. *Any students who do not make good use of office hours and who also perform very badly in a midterm exam will be awarded a black mark. Any students who are absent from their autopsies — so to speak — will be awarded a black mark.*

**Makeup Exams:** If any are necessary, they are to be held at the end of the semester. Since group theory is a “click topic” — hard at first but becoming easy with practise — the Midterm Makeups will be significantly harder than the Midterms. *And the black marks still apply.*

**Availability:** At the beginning of the course, you must give me your e-mail address so that, if necessary, I can summon you to the Tuesday Office Consultation Hours, giving you 24 hours notice. *Any students who do not give me their e-mail address, or who do not respond to a summons, will be awarded a black mark.*

**Minimal achievement threshold:** A basic competence in group theory includes, for instance: an ability to find all the subgroups of small groups such as $S_3$, $D_8$, $Q_8$, $A_4$; an ability to state and prove the easier theorems such as Lagrange’s Theorem and the Three Isomorphism Theorems; the ability to state and apply Sylow’s Theorem. Any students who lack such abilities by the end of the course would find *Algebra II* impossible. They will have to be given an F to make them ineligible for that course. This is for the sake of the genuine students, who will have earned the right to study Galois theory in a zombie-free environment.

**Grading algorithm:** Curve method. Step 1: After making a survey of average GPAs previously achieved for this and similar courses, I decide the approximate average GPA for the class; this depends largely on mathematical achievement, but partly also on maturity of academic behaviour; participation, disruption, attendance. Step 2: I assign grades to clusters of credit unit scores. Step 3: For students with a black mark and a grade that is not already F, I reduce their grade by one level, for instance, C to C- or C- to D+. Multiple black marks merit multiple reductions of the grade. Step 5: I calculate the average GPA. If it is very close to the decided approximate average, the process terminates, otherwise I return to Step 2.
What are group theory and algebra?

To begin with the obvious, group theory, and algebra in general, are human activities, components of human culture. To describe any component of human culture, one cannot describe its future, because that is unknown. One cannot describe its present state, because that is just a phantom which flickers for a single moment and then immediately becomes the past. So one can only describe the past. Even in research, all the resources — knowledge, outlooks, aspirations — are drawn from the history of the topic. When mathematicians debate, say, the quality of some research work or the potential for some proposed research programme, what data do they use to support their arguments? They appeal to precedents and paradigms. They may, for instance, make comparisons with previous work that has turned out to be successful and productive. In other words, their debates are nothing more nor less than debates in applied history. So, to answer the eponymous question, we must give a brief account of the history of group theory and algebra.

Algebra is the name of a particular current of thought which, up until the 19th century, was largely concerned with the problem of finding the solutions to polynomial equations

$$a_nX^n + a_{n-1}X^{n-1} + ... + a_1X + a_0 = 0.$$ 

That current of thought can be traced back through the mathematics of the Babylonians, the classical Greeks, the medieval Hindus, the medieval Arabs. The term algebra derives from the title of a 9th century Arabic work on quadratic equations, that is to say, the case $n = 2$. Or rather, since the modern notion of an equation had not yet been developed, we should say that the material in that Arabic work was equivalent to a theory of quadratic equations.

Cardan’s *Ars Magna*, 1545, presented methods for solving for the cubic and quartic equations, that is, the cases $n = 3$ and $n = 4$. The notion of a formula rather than an algorithm gradually emerged, largely though the work of Vieta in the late 16th century. Vieta, feeling that the word *algebra* might sound too eastern for Christian tastes, proposed the alternative term *analysis*.

During the early 17th century, the words *algebra* and *analysis* had much the same meaning, though differing perhaps in nuance. Several writers stated, without proof, the Fundamental Theorem of Algebra. In modern terminology: every polynomial equation with coefficients $a_0$, ..., $a_n$ in $\mathbb{C}$ has a solution in $\mathbb{C}$.

Following the advent of differential and integral calculus, pioneered by Fermat, Descartes, Newton, Leibniz and others, a distinction came to be recognized between “ordinary algebra” (involving just addition, multiplication, subtraction, division, extraction of $n$-th roots) and “infinite analysis” (involving also the more subtle operation of taking limits).

For a long while, scant progress was made in “ordinary algebra”. No formula was found for the quintic equation, the case $n = 5$. In 1771, Lagrange made use of symmetry arguments to unify the treatment of the cubic and quartic. He noticed that the approach does not work when applied to the quintic.

Beginning in 1799, Ruffini started working towards a proof that the solutions to the quintic cannot be expressed by means of a formula involving the five operations of “ordinary algebra”. His best effort, in 1813, is nowadays regarded as essentially valid, although the earliest proof recognized by contemporaries was by Abel, in 1824. Both proofs, again, made use of symmetry arguments.

In about 1830, Galois obtained a slight improvement of the Abel–Ruffini Theorem. He proved that there exist quintic polynomials whose roots cannot be expressed using the five operations of “ordinary algebra”. One of his key concepts was an abstract device for expressing
the concept of symmetry. He called it a *group*; in English translation, a *group*. The Galois group of a polynomial equation is the set of symmetries of the equation. The intricacy of the structure of the Galois group can be taken as a measure of the intricacy of the calculations associated with the equation. But he was not interested in methods for doing the calculations. His aim was to describe something deeper than lies behind the calculations. The following quote, in English translation, is from a draft of a preface for a book that he intended to produce.

“Jump on calculations with both feet; group the operations, classify them according to their difficulty and not according to their form; such, according to me, is the task of future geometers; such is the path that I have embarked upon in this work. ... There will be time to carry out the calculations envisioned by this high analysis, and classified according to their difficulty but not specified by their form, when the details of a question reclaim them.”

He used the incongruous word “geometers” because, at that time, *algebra* was just the elementary skill of manipulating ordinary equations without calculus or infinite sums or any kind of limit operation. Algebra was not yet seen as a field of research.

Let us make a few technical comments to give an indication of the idea behind Galois theory. For simplicity of discussion, let us consider only those polynomial equations $a_nX^n + ... + a_0 = 0$ such that the coefficients $a_n, ..., a_0$ are rational numbers. Let us also confine our attention to the irreducible case, we mean to say, the case where left-hand expression cannot be written as a product of two non-constant factors $(b_nX^n + ... + b_0)(c_lX^l + ... + c_0)$ with rational coefficients. An example is the quadratic equation $X^2 + 1 = 0$. The two solutions $X = i$ and $X = -i$ are, in some algebraic sense, indistinguishable; they are interchanged by a symmetry operation, namely, the complex conjugation operation $x + iy \mapsto x - iy$, which preserves addition and multiplication. Another example is the equation $X^2 - 2 = 0$. Again, two solutions $X = \sqrt{2}$ and $X = -\sqrt{2}$ are algebraically indistinguishable. This time, the symmetry interchanging them is given by $x + y\sqrt{2} \mapsto x - y\sqrt{2}$, which does preserve addition and multiplication provided we insist that $x$ and $y$ are rational numbers. In both cases, there are actually two symmetry operations: one of them being the operation $x + iy \mapsto x - iy$ or $x + y\sqrt{2} \mapsto x - y\sqrt{2}$, the other being the identity operation $x + iy \mapsto x + iy$ or $x + y\sqrt{2} \mapsto x + y\sqrt{2}$. In an abstract way, the two quadratic equations $X^2 + 1 = 0$ and $X^2 - 2 = 0$ have the same features. For both equations, the Galois group expressing the symmetries is a group called $C_2$, which has size 2.

The situation becomes a little more complicated for irreducible cubic equations. Let us illustrate the phenomena that arise by discussing two examples. First, consider the cubic equation $X^3 - 2 = 0$. The three solutions are $t$ and $\omega t$ and $\omega^2 t$ where $t$ is the real cube root of 2 and $\omega = e^{2\pi i/3}$. It is easy to show that $t$ is irrational. So $X^3 - 2$ has no rational solutions and it cannot be written as the product $(b_2X^2 + b_1X + b_0)(c_1X + c_0)$. We have shown that the equation $X^3 - 2 = 0$ is irreducible.

The three roots $t$ and $\omega t$ and $\omega^2 t$ are algebraically indistinguishable. That may seem to be a peculiar assertion, since one of the roots is real and the other two are not real. But still, it is impossible to distinguish between the roots just in terms of the “ordinary algebra” of the rational numbers. In the *Algebra II* course, we shall see that the Galois group of the equation $X^3 - 2 = 0$ is a group called $S_3$, which has size 6. That is to say, all six possible permutations of the three roots preserve addition and multiplication. The equation has precisely 6 symmetry
operations. Some theoretical technology is needed to make that assertion precise, and to prove it. For the moment, let us just note that one of the non-identity symmetries is given by complex conjugation, which fixes the solution \( t \) and interchanges the solutions \( \omega t \) and \( \omega^2 t \).

The first interesting surprise in Galois theory is that not all irreducible cubics have the same underlying symmetry. Our other example is the cubic equation

\[ X^3 + X^2 - 2X - 1 = 0. \]

There is no rational root \( p/q \), since otherwise, assuming that the non-zero integers \( p \) and \( q \) are as small as possible, we would have \( p^3 + p^2q - 2pq^2 - q^3 = 0 \), hence every prime divisor of \( p \) would divide \( q \), and every prime divisor of \( q \) would divide \( p \), yielding \( p/q = \pm 1 \), which is absurd. So, as before cubic expression \( X^3 + X^2 - 2X - 1 \) cannot be factorized as the product of a quadratic expression and a linear expression. We have shown that the equation is irreducible.

We claim that the three solutions are \( X = c_1 \) and \( X = c_2 \) and \( X = c_3 \) where

\[
\begin{align*}
c_1 &= \zeta + \zeta^5 = 2 \cos(2\pi/7) = 2 \cos(12\pi/7), \\
c_2 &= \zeta^2 + \zeta^3 = 2 \cos(4\pi/7) = 2 \cos(10\pi/7), \\
c_3 &= \zeta^3 + \zeta^4 = 2 \cos(6\pi/7) = 2 \cos(8\pi/7),
\end{align*}
\]

and \( \zeta = e^{2\pi i/7} \). To see this, first observe that \( \zeta^7 = 1 \) and

\[
(1 + \zeta + \zeta^2 + ... + \zeta^6)(1 - \zeta) = 1 - \zeta^7 = 0
\]

hence \( \zeta + \zeta^2 + ... + \zeta^6 = -1 \). We have

\[
c_1c_2 + c_2c_3 + c_1c_3 = (\zeta + \zeta^5)(\zeta^2 + \zeta^3) + (\zeta^2 + \zeta^5)(\zeta^3 + \zeta^4) + (\zeta + \zeta^7)(\zeta^3 + \zeta^4)
\]

\[
= 2(\zeta + \zeta^2 + \zeta^3 + \zeta^4 + \zeta^5 + \zeta^6) = -2.
\]

By a similar method, it is easy to show that \( c_1 + c_2 + c_3 = -1 \) and \( c_1c_2c_3 = 1 \). Therefore

\[
(X - c_1)(X - c_2X - c_3) = X^3 - (c_1 + c_2 + c_3)X^2 + (c_1c_2 + c_2c_3 + c_1c_3)X + c_1c_2c_3
\]

\[
= X^3 + X^2 - 2X - 1.
\]

The claim is established.

Once again, the solutions are algebraically indistinguishable. None of them can be distinguished from the others in terms of the “ordinary algebra” of the rationals. There is the identity symmetry, which replaces \( c_1, c_2, c_3 \) with \( c_1, c_2, c_3 \), respectively. There is another symmetry, replacing \( c_1, c_2, c_3 \) with \( c_2, c_3, c_1 \), respectively. And there is one more symmetry, replacing \( c_1, c_2, c_3 \) with \( c_3, c_1, c_2 \). But, this time, there are no other symmetries. We cannot interchange two of the three solutions, leaving the other one fixed. Indeed, using the method of the previous paragraph, or using the formula \( \cos(2\theta) = 2\cos^2(\theta) - 1 \), we see that three solutions are related by

\[
c_2 = c_1^2 - 2, \quad c_3 = c_2^2 - 2, \quad c_1 = c_3^2 - 2.
\]

These three equations belong to “ordinary algebra” and their coefficients are rational numbers. If we were to interchange two of the roots, leaving the other one fixed then, in a manner of speaking, the latest three equations would crash; the substitution can be detected in terms of
the “ordinary algebra” of the rationals. In the language of Galois theory, the Galois group of the equation \( X^3 + X^2 - 2X - 1 = 0 \) is a group called \( C_3 \), which has order 3.

In the above discussion, we did a lot of calculation with a particular example. Calculations with examples are necessary in algebra, as a way of gaining insights into the underlying principles. But Galois’ stated aim was to transcend the particular calculations. In the *Algebra II* course, we shall prove that every irreducible cubic equation with rational coefficients has Galois group \( S_3 \) or \( C_3 \), and moreover, upon reducing to the form \( X^3 + cX + d = 0 \) via a linear substitution, the Galois group is \( C_3 \) if and only if the square roots of \(-4c^3 - 27d^2\) are rational numbers.

All irreducible quartics over the rationals have Galois groups \( C_4 \) or \( V_4 \) or \( A_4 \) or \( S_4 \), which have orders 4 and 4 and 12 and 24. For irreducible quintics over the rationals, the possible Galois groups are \( C_5 \), \( D_{10} \), \( A_5 \), \( S_5 \), which have orders 5, 10, 60, 120. But here, another surprise appears. Some trouble arises in the case where the Galois groups are \( A_5 \) and \( S_5 \). Most small groups have a property called solvability: roughly, that means that they can be built up from a group with size 1 in a series of steps, a single cycle of symmetries being adjoined at each step. Galois proved that a polynomial equation is solvable in the sense of “ordinary algebra” if and only of the Galois group is solvable in the sense that we have just indicated. All the Galois groups that can arise for quadratic, cubic or quartic equations are solvable, and so too are the groups \( C_5 \) and \( D_{10} \) but, in *Algebra I*, we shall prove that \( A_5 \) and \( S_5 \) are not solvable.

By the time of Galois, the term “infinite analysis” had been contracted to *analysis*. It was a discipline with two main branches: complex analysis and Fourier analysis. But the term *algebra* was used only in reference to the elementary art of manipulating ordinary equations without using calculus or infinite sums or any kind of limiting process; *algebra* was not yet the name of any area of mathematical research.

Galois’ work publicized only in 1843, by Liouville, who had made a study of some manuscripts that Galois had left behind. Even after that, for a long while, the work received very little attention. One reason for this may have been that Liouville promised to give a more detailed account of the theory at a later date, but he never fulfilled that promise. Mathematicians tend to avoid a line of research if there are rumours that someone else is about to clean it out. One mathematician who did pay attention to the new topic was Sylow. Eventually, in 1872, he was to prove his celebrated theorem, which is one of the highlights of the *Algebra I* course. But he was delivering lecture courses on the topic long before that. One of the students attending his group theory lectures in 1862 was Lie. Actually, though, Lie later confessed that he had understood very little of those lectures.

During the 1870s, Lie teamed up for a little while with Klein, a geometer. One of Klein’s big ideas, the *Erlanger Programme*, proposed in 1872, was that geometry should be understood as the study of features that are preserved by the underlying symmetries of a space. In 1873, Lie and Klein were discussing the possibility of some kind of Galois theory for differential equations. Such a theory was developed by Picard in the 1880s and Vessiot in the early 1890s. Meanwhile, Lie was making a study of the underlying symmetry groups, infinite and continuous, nowadays called Lie groups.

To complete this outline of the emergence of algebra in the 19th century, one other very important line of research has to be mentioned. In 1815, motivated by a study of polynomial equations, Cauchy introduced a theory of “substitutions”. It was group theory, in a way, except that Cauchy, like Abel, had not taken the step of fully abstracting the notion from its context.
Cauchy returned to his “substitutions” during the period 1844 to 1847, in his efforts to prove a unique factorization conjecture that implied Fermat’s Last Theorem.

Meanwhile, starting in 1837, Kummer was making a study of higher reciprocity laws in number theory. In 1847, Cauchy received the news that Kummer had refuted the desired unique factorization property. Discouraged, Cauchy abandoned the work and turned to other topics. But, during the next four decades, Kummer’s ideas inspired Kronecker, who did much to initiate algebraic geometry, and they also inspired Dedekind, who did much to clarify Galois theory and to develop it in the direction of what is nowadays called algebraic number theory.

Several intertwining threads of algebraic research emerged during the 1890s, each of them having such a heavy influence on the others that it becomes impossible to disentangle them for the purpose of contriving a single line of historical narrative. So let us confine our attention, now, to just one particular topic, group representation theory (which I select because it happens to be my own speciality).

Group representation theory can be described as a study of the symmetries of mathematical objects that have some kind of linear structure, vector spaces especially. On the one hand, the theory has many applications, because linear structures arise frequently throughout mathematics. On the other hand, linear structures tend to have rich features, and much use of this can be made in the abstract study of groups themselves.

Even if we are to describe just the origins of representation theory, there are several threads to be disentangled. In connection with Galois theory, in about 1880, Dedekind had made a study of a now obsolete concept called the “group determinant”. That led Frobenius, starting in the late 1890s, to pioneer a study of the way the elements of finite groups can be “represented” by matrices; hence the name of topic. But this approach to the study of Lie groups had begun about ten years before that. Lie and his many followers, adopting the outlook of Klein’s Erlanger Programme, had regarded his groups as “transformation groups”, that is, groups which preserve geometrical structure. Groups representations in the linear sense had been heavily involved in that. Also in the late 1880s, Hilbert recognized that a topic called the theory of invariants could be generalized and incorporated into this group theoretical approach to geometry. Burnside, starting from the point of view of invariant theory, began to investigate the finite symmetries of inner product spaces.

Let us mention one early application of representation theory as a tool in finite group theory: Burnside’s $p^\alpha q^\beta$ Theorem asserts that, given a finite group $G$ with size $|G| = p^\alpha q^\beta$, where $p$ and $q$ are prime, then $G$ is solvable. In other words, the size of any non-solvable finite group has at least 3 distinct prime factors. For example, the non-solvable groups $A_5$ and $S_5$ have sizes 60 and 120, the prime factors being 2, 3, 5. In this connection, let us also mention a much later application of representation theory: The Odd Order Theorem, proved by Feit and Thompson in 1963, asserts that, given a finite group $G$ such that $|G|$ is odd, then $G$ is solvable. Their very densely-written proof took up an entire issue of a journal. An alternative proof of the theorem would lead to instant fame. (Part of their argument would be considerably simplified by a proof of the Feit–Thompson Conjecture: there do not exist distinct primes $p$ and $q$ such that $(p^q - 1)/(p - 1)$ divides $(q^p - 1)/(q - 1)$.)

By the beginning of the 20th century, the structural approach to pure mathematics had become fully established, and algebra too, had become fully established as one of the main branches of pure mathematics. We have have defined the term by an appeal to the history: algebra is the name for mathematics, pure or applied, that has it roots in the 19th century work that we have indicated above.

We ought to say a few words about the impact of algebra beyond the realm of pure math-
Let us just mention some of the earliest applications of group representation theory to physics. During the 1920s, finite group representation theory acquired a minor application to chemistry, in the study of modes of vibration of molecules. But some more spectacular applications appeared in quantum mechanics during the late 20s and early 30s. Dirac’s relativistic quantum mechanics made much use of the representations of the Lie group $U(2)$, the group of $2 \times 2$ unitary matrices, which is a “double cover” of $SO(3)$, the group of rotations of 3-dimensional Euclidian space. In a certain sense, some quantum systems (for instance, electrons orbiting a nucleus) are returned to their original state upon rotation through an angle of $4\pi$, not $2\pi$. Finite group representation theory appears too, one of the reasons being that, up to a bizarre $\pm 1$ factor, quantum systems are symmetrical under permutations of indistinguishable particles (for instance, if two of the orbiting electrons are interchanged with each other, then the vector representing the state of the system is multiplied by $-1$.)

And, just in case those applications seem too abstruse, let us add that this kind of mathematical physics nowadays has applications to engineering. The variables that quantum physicists see as position and momentum have been reinterpreted, by signal-processing engineers, as time and frequency. One of the symmetry groups of this system turns out to be the group of area-preserving transformations of the Euclidian plane. That group includes time and frequency shifts. It includes the fractional Fourier transform, which is used to reconstruct signals from certain kinds of samples, for instance, in medical tomography. It also includes chirp transforms, corresponding to certain shears of the plane; the name comes from the fact that these transforms convert a monotone note into a sound rather like the chirp of a bird.

Syllabus

- Some ring theory...

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