10 January 2007, LJB, Bilkent University.

Time allowed: 2 hours.

Each question is worth 25% of the available marks.

Please make sure your name is on every sheet of your script.

1: Let $p$ be a prime. Prove that every finite group has a Sylow $p$-subgroup.

2: Let $p$ be an odd prime, let $a$ be a positive integer, let $q = p^a$ and let $F$ be a field with order $q$. The group $GL_2(F)$ is the group of invertible $2 \times 2$ matrices with entries in $F$. The group $SL_2(F)$ is the subgroup consisting of those matrices which have determinant 1. The group $PGL_2(F)$ is defined to be the quotient group $PGL_2(F) = GL_2(F)/Z(GL_2(F))$. For each of these groups, find its order, determine the Sylow $p$-subgroups up to isomorphism, and find the number of Sylow $p$-subgroups.

3: (a) Prove that the alternating group $A_5$ is simple.

(b) For each $n \in \{1, 2, 3, 4, 5, 6\}$, let $G_n$ be the stabilizer of $n$ in $A_6$. Show that the subgroups $G_n$ are isomorphic to $A_5$ and that they are mutually conjugate.

(c) Prove that $A_6$ is simple by considering a normal subgroup $N \trianglelefteq A_6$ and by showing that $G_n \leq N$ or $G_n \cap N = 1$ for each $n$.

4: In the two $4 \times 4$ squares below, each entry has at most four adjacent entries. For example, in both squares, 10 is adjacent to 6, 9, 11, 14. Using the theory of the symmetric and alternating groups, prove that it is impossible to get from the first square to the second square by a sequence of operations where each operation interchanges the $\bullet$ entry with an adjacent entry.

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\begin{array}{cccc}
2 & 1 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & \bullet \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & \bullet \\
\end{array}
\]
10 November 2006, LJB, Bilkent University.

Time allowed: 110 minutes.

Attempt FIVE questions. Or, if you attempt all six, then the best five will be counted. Each question carries 20% of the marks.

Please make sure your name is on every sheet of your script.

A1: Define the following terms: (a) group, (b) subgroup, (c) normal subgroup. (You may assume that the reader is familiar with general set-theoretic terminology, such as binary operation, but non-set-theoretic terminology and notation must be specified clearly.)

A2: Give either a proof or a counter-example for the following assertion: “Given a subgroup $H$ of a group $G$, then there are only finitely many left cosets of $H$ in $G$ if and only there are only finitely many right cosets of $H$ in $G.”

A3: State and prove the Third Isomorphism Theorem for Groups. (Any version of the theorem will be acceptable.)

A4: Let $G$ be the group with order 21 generated by elements $a$ and $b$ such that $a^7 = b^3 = 1$ and $ba = a^2b$. (You may assume that such a group exists.)

(a) Find all the normal subgroups of $G$.

(b) Up to isomorphism, find all the groups $F$ such that there exists a surjective homomorphism from $G$ to $F$. (Hint: Use the First Isomorphism Theorem.)

A5: Let $G$ be a finite group, and let $x$ be an element of a transitive $G$-set $X$. Let $\text{Stab}_G(x)$ denote the stabilizer of $x$ in $G$. State and prove the Orbit-Stabilizer Equation for $|\text{Stab}_G(x)|$.

A6: For a positive integer $n$, the $n$-cube (the $n$-dimensional cube) has $2^n$ vertices. The vertices can be identified with the $n$-tuples $(a_1, ..., a_n)$ where each $a_i \in \{-1, 1\}$. There is an edge between the vertices $(a_1, ...)$ and $(b_1, ...)$ when $a_i = b_i$ for all except one of the indices $i$.

(a) Let $G$ be the group of rigid symmetries of the $n$-cube. What is the order of $G$?

(b) Show that the $n$-cube has precisely $2^{n-1}n$ edges.

(c) Let $\epsilon$ be an edge. Find the order of the group $\text{Stab}_G(\epsilon)$. (Hint: Obviously, the edges of the $n$-cube comprise a transitive $G$-set.)
1: What is a Sylow $p$-subgroup of a finite group? You may assume the first part of Sylow’s Theorem, that is to say, the existence part, which asserts that any finite group has a Sylow $p$-subgroup. State and prove the rest of Sylow’s Theorem.

2: Let $P$ be a Sylow $p$-subgroup of the symmetric group $G = S_p$.

(a) Express $|C_G(P)|$ and $|N_G(P) : P|$ in terms of $p$. (Hint: recall that the group $\text{Aut}(C_p) \cong (\mathbb{Z}/p)^* \text{ has order } p - 1$.)

(b) Wilson’s Theorem, in classical number theory, asserts that $(p - 1)! + 1$ is divisible by $p$. Prove it using part (a) and Sylow’s Theorem. (No marks will be awarded for other proofs.)

3: Let $n$ be a positive integer. You may assume that, for any abelian group $A$ with order $p^n$, there exist positive integers $n_1 \geq \ldots \geq n_r$ such that $n_1 + \ldots + n_r = n$ and

$$A \cong \mathbb{Z}/p^{n_1} \times \ldots \times \mathbb{Z}/p^{n_r}.$$  

(a) Let $\Omega(A)$ be the subgroup of $A$ consisting of the elements with order 1 or $p$. In other words, $\Omega(A) = \{a \in A : pa = 0\}$. Express $|\Omega(A)|$ in terms of $p$ and $r$.

(b) Express $A/\Omega(A)$ as a direct product of cyclic groups.

(c) Let $m_1 \geq \ldots \geq m_s$ be positive integers such that $m_1 + \ldots + m_s = n$, and let $B$ be a group such that

$$B \cong \mathbb{Z}/p^{m_1} \times \ldots \times \mathbb{Z}/p^{m_r}.$$  

Using parts (a) and (b), show that $A \cong B$ if and only if $r = s$ and each $n_i = m_i$.

(d) Up to isomorphism, how many abelian groups with order 1024 are there?

4: (a) How many conjugacy classes does $S_{10}$ have?

(b) How many of those conjugacy classes belong to the alternating group $A_{10}$?

(c) Show that, for any positive integer $n$, the number of conjugacy classes of $S_n$ is equal to the number of isomorphism classes of abelian groups with order $p^n$. (You may assume all the material in Question 3.)
A1: (a) Define the terms group and abelian group

(b) Given examples of: (b1) a finite abelian group, (b2) a finite non-abelian group, (b3) an infinite abelian group, (b4) an infinite non-abelian group. (For each example, define the group that you are specifying; do not just give its name.)

A2: Let $H$ be a subgroup of a group $G$.

(a) Show that $G$ is the disjoint union of the left cosets of $H$ in $G$.

(b) State and prove Lagrange’s Theorem.

A3: The group $Q_8$ can be defined as the group generated by the elements

$$x = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

(a) Show that $Q_8 = \{1, -1, x, -x, y, -y, z, -z\}$ where $z = xy$ and 1 denotes the identity matrix.

(b) Find all the subgroups of $Q_8$.

(c) Which of the subgroups are normal?

A4: Let $G$ be an abelian group and let $E \leq F \leq G$. Using additive notation throughout, state and prove the Third Isomorphism Theorem for $E, F, G$. (Only half-credit will be awarded for a correct statement and proof using multiplicative notation.)

A5: Let $G$ be a group and let $X$ and $Y$ be transitive $G$-sets. Let $x \in X$ and $y \in Y$. Let $A = \text{Stab}_G(x)$ and $B = \text{Stab}_G(y)$.

(a) Show that $fA = \text{Stab}_G(fx)$ for all $f \in G$.

(b) Show that $A$ and $B$ are conjugate subgroups if and only if there exists a bijection $\theta : X \to Y$ such that $\theta(gx) = y\theta(x)$ for all $g \in G$. (Hint: if $B = fA$ with $f \in G$, first replace $x$ with $fx$ to reduce to the case where $A = B$, then show that there is a well-defined $\theta$ given by $\theta(gx) = gy$.)
1: For finite groups, one way of expressing the Direct Product Recognition Theorem is as follows: given normal subgroups $H$ and $K$ of a finite group $G$, and supposing that $H \cap K = \{1\}$ and $|H| \cdot |K| = |G|$, then $G \cong H \times K$. State and prove a version of the theorem that also applies to infinite groups.

2: (a) For each of the primes $p \in \{3, 5, 7\}$, find the number of Sylow $p$-subgroups in the alternating group $A_7$.

(b) Let $P$ be the subgroup of $A_7$ generated by the elements $a = (1, 2, 3, 4)(5, 6)$ and $b = (1, 3)(2, 4)$. Show that $P$ is isomorphic to the dihedral group $D_8$. Deduce that $P$ is a Sylow 2-subgroup of $A_7$.

(c) Find the number of Sylow 2-subgroups of $A_7$.

3: A Germain prime is a prime $p$ such that the integer $q = 2p + 1$ is prime too. For such $p$ and $q$, let $G$ be a non-abelian group with order $pq$. Find the number of conjugacy classes in $G$.

4: The icosahedral group $Icos$ is defined to be the group of rigid symmetries of an icosahedron. (It is also the group of rigid symmetries of a dodecahedron. It is generated by the rotations and the reflections.)

(a) Show that $Icos$ has the same order as the symmetric group $S_5$.

(c) Show that $Icos$ is not isomorphic to $S_5$. (Hint: consider the element of $Icos$ that sends each point of the icosahedron to its opposite point.)

5: Find all the conjugacy classes of $S_5$. Determine the size of each conjugacy class, and check that the sum of their sizes is equal to the order of $S_5$. 